On the simple connectedness of certain subsets of buildings

Alice Devillers* Bernhard Mühlherr†

October 4, 2005

Université Libre de Bruxelles
Département de Mathématiques - C.P.216
Boulevard du Triomphe
B–1050 Brussels, Belgium

Abstract

We prove a rank 3 criterion for the simple connectedness of certain subsets of buildings and we give two applications of this criterion. The first generalizes a result of Tits for Chevalley groups to 3-spherical Kac-Moody groups. The second is the proof of the simple connectedness of certain flipflop geometries introduced in [BGHS].

MSC (2000): 20E42, 20G40, 51E24

1 Introduction

Chamber systems have been introduced by J. Tits in [Ti81] in order to study buildings from a 'local' point of view. He showed that buildings are precisely the simply connected chamber systems of Coxeter type which satisfy an additional rank 3 criterion. It turned out that chamber systems and their connectivity properties are useful in studying amalgam presentations of groups (see [Ti86]). The principal observation is the so called Tits lemma. It says that a group acting transitively on the set of chambers of a simply connected chamber system is the amalgam of the stabilizers of certain of its residues.

Using this observation there is a new interpretation of certain amalgam presentations of finite Chevalley groups which had been considered by K.-W. Phan in [Ph77]. These amalgam presentations are of some interest in the classification of the finite simple groups. This fact provided the motivation for the authors of [BGHS] to revise and improve Phan's results. In [BGHS], they describe the general ideas of their 'Phan-program'. They introduce flips of spherical buildings and twin buildings. Using the Tits lemma they observe that Phan's results are actually equivalent to the simple connectedness of certain flipflop

*Chargé de Recherche of the Fonds National de la Recherche Scientifique (Belgium); e-mail: adevil@ulb.ac.be

†e-mail: bmuhlher@ulb.ac.be

geometries. Up to now they proved in a series of papers the simple connectedness of the relevant flipflop geometries for the groups of type A_n, B_n, C_n and D_n. This is done by studying the corresponding building via the natural modules. It is however not clear to which extent these methods are suitable to handle the exceptional types as well.

The main result of the present paper is motivated by this problem. We provide a rank 3 criterion for the simple connectedness of the flipflop geometry associated with a flip of a twin-building. This is our Theorem 6.6. This criterion reduces the problem of handling the exceptional Chevalley groups to a careful investigation of flips of Chevalley groups of rank at most 3. A considerable part of this analysis has already been carried out in the papers mentioned above. A detailed and complete analysis of these rank 3 conditions will be given in [GHMS] where Phan's results will be revised and improved for Chevalley groups and generalized to Kac-Moody groups.

The principal idea of the present paper is to introduce *filtrations* of chamber systems in order to prove the simple connectedness of some of their subsystems. This technique is well known for simplicial complexes and has been used by P. Abramenko in order to study groups acting on twin buildings in [Ab95]. In Section 3 we give a general rank 3 criterion for the simple connectedness of the chamber system induced on a subset of chambers of a simply connected chamber system using filtrations. This is our Theorem 3.14. As indicated above, the main motivation for proving this rank 3 criterion is provided by the revision of Phan's theorem. We will apply our criterion to one half of a twin building associated to a Kac-Moody group in order to obtain a rank 3 criterion for the simple connectedness for the flipflop geometries, which is our Theorem 6.6 already mentioned. Unfortunately, the details of the application of this criterion in order to obtain Phan's original results are too technical to be given here. They depend heavily on the the local analysis which will be carried out in [GHMS].

Nevertheless, there is another application of our general criterion to the chamber system far away from a chamber in a twin building. It is the following theorem which is a reformulation of Theorem 5.2 below. We refer to Section 4 and 5 for the definitions and notation.

Theorem 1.1.: *Let $\mathcal{B}$ be a 3-spherical twin building and let c be a chamber of $\mathcal{B}$. Then the chamber system induced on c^{op} is simply 2-connected if this is true for all rank 3 residues containing c and if for each rank 2 residue R containing c the set of chambers opposite to c in R is connected.*

The theorem above is a generalisation of a result of Tits in the spherical case ([Ti86], Proposition 7). However, our methods of proving it are different from those used in loc. cit.. The proof of [Ti86] relies heavily on the finiteness of the root system and it is not clear how to make its arguments work without the finiteness assumption. Thus, our paper provides a different approach in the spherical case and a new result in the Kac-Moody situtation. Using results of [Ti86] and [Ab95] one obtains the following as a consequence of the theorem above.

Corollary 1.2.: *Let G be a 3-spherical Kac-Moody group over a field F having at least 16 elements. Let B_+, B_- be opposite Borel subgroups and for $\epsilon \in \{+,-\}$ let P_s^ϵ be the* *minimal parabolic subgroups containing B_ϵ. Then B_+ is the 2-amalgam of the groups $(B_+ \cap P_s^-)_{s\in S}$. In particular, B_+ is finitely presented if F is finite.*

We remark that the finite presentability of B_+ has been announced in [Ab02].

2 Chamber Systems

Let I be a set. A *chamber system* over I is a pair $\mathcal{C} = (C, (\sim_i)_{i\in I})$ where C is a set whose elements are called *chambers* and where $\sim_i$ is an equivalence relation on the set of chambers for each $i \in I$, such that if $c \sim_i d$ and $c \sim_j d$ then either $i = j$ or $c = d$. Given $i \in I$ and $c, d \in C$, then c is called *i-adjacent* to d if $c \sim_i d$. The chambers c, d are called *adjacent* if they are i-adjacent for some $i \in I$.

For the rest of this subsection let $\mathcal{C} = (C, (\sim_i)_{i\in I})$ be a chamber system over I. A *gallery* in $\mathcal{C}$ is a finite sequence $(c_0, c_1, \ldots, c_k)$ such that $c_\mu \in C$ for all $0 \le \mu \le k$ and such that $c_{\mu-1}$ is adjacent to c_μ for all $1 \le \mu \le k$. The number k is called the *length* of the gallery. Given a gallery $G = (c_0, c_1, \ldots, c_k)$, then we put $\alpha(G) = c_0$ and $\omega(G) = c_k$. If G is a gallery and if $c, d \in \mathcal{C}$ such that $c = \alpha(G), d = \omega(G)$, then we say that G is a *gallery from c to d* or *G joins c and d*. The chamber system $\mathcal{C}$ is said to be *connected* if for any two chambers there exists a gallery joining them. A gallery G will be called *closed* if $\alpha(G) = \omega(G)$. A gallery $G = (c_0, c_1, \ldots, c_k)$ will be called *simple* if $c_{\mu-1} \neq c_\mu$ for all $1 \le \mu \le k$.

Given a gallery $G = (c_0, c_1, \ldots, c_k)$ then G^{-1} denotes the gallery $(c_k, c_{k-1}, \ldots, c_0)$ and if $H = (c'_0, c'_1, \ldots, c'_l)$ is a gallery such that $\omega(G) = \alpha(H)$, then GH denotes the gallery $(c_0, c_1, \ldots, c_k = c'_0, c'_1, \ldots, c'_l)$.

Let J be a subset of I. A *J-gallery* is a gallery $G = (c_0, c_1, \ldots, c_k)$ such that for each $1 \le \mu \le k$ there exists an index $j \in J$ with $c_{\mu-1} \sim_j c_\mu$. Given two chambers c, d, then we say that c is *J-equivalent* to d if there exists a J-gallery joining c and d and we write $c \sim_J d$ in this case. Note that c, d are i-adjacent if and only if they are $\{i\}$-equivalent. Given a chamber c and a subset J of I then the set $R_J(c) := \{d \in C \mid c \sim_J d\}$ is called the *J-residue* of c. The pair $\mathcal{R}_J := (R_J(c), (\sim_j)_{j\in J})$ is a connected chamber system over J which is also called the J-residue of c. If $J = \{i\}$, then $R_J(c)$ is called the *i-panel* of c (or the i-panel containing c); a *panel* is an i-panel for some $i \in I$.

Homotopy of Galleries and Simple Connectedness

Throughout this subsection let $m \ge 1$ be a natural number and let $(\mathcal{C}, (\sim_i)_{i\in I})$ be a chamber system over a set I.

Given two galleries G, H then $G = (c_0, \ldots, c_k)$ and $H = (c'_0, \ldots, c'_{k'})$ are said to be *elementary m-homotopic* if there exist $0 \le \mu \le \nu \le k, 0 \le \mu' \le \nu' \le k'$ such that the following holds:

(H1) $\mu = \mu'$ and $c_\eta = c'_\eta$ for all $0 \le \eta \le \mu$.

(H2) $k - \nu = k' - \nu'$ and $c_{k-\eta} = c'_{k'-\eta}$ for all $0 \le \eta \le k - \nu$.

(H3) The galleries $(c_\mu, \ldots, c_\nu), (c'_{\mu'}, \ldots, c'_{\nu'})$ are J-galleries for some subset J of I having cardinality at most m.

Two galleries G, H are said to be *m-homotopic* if there exists a finite sequence $G_0, G_1, \ldots, G_l$ of galleries such that $G_0 = G, G_l = H$ and such that $G_{\mu-1}$ is elementary m-homotopic to G_μ for all $1 \leq \mu \leq l$.

If two galleries G, H are m-homotopic, then it follows by the definition that $\alpha(G) = \alpha(H)$ and $\omega(G) = \omega(H)$. A closed gallery G is said to be *null-m-homotopic* if it is m-homotopic to the gallery $(\alpha(G))$. The chamber system $\mathcal{C}$ is called *simply m-connected* if it is connected and if each closed gallery is null-m-homotopic.

We close this section by stating some elementary observations concerning homotopies.

1. Given a gallery G, then GG^{-1} is null-m-homotopic.

2. Two galleries H, G are m-homotopic if and only if the gallery GH^{-1} is null-m-homotopic.

3 A rank 3-criterion for simple 2-connectedness

Throughout this section let I be a set and let $\mathcal{C} = (C, (\sim_i)_{i\in I})$ be a chamber system over I.

A *filtration* of $\mathcal{C}$ is a family $\mathcal{F} = (C_n)_{n\in\mathbf{N}}$ of subsets of C such that the following holds.

(F1) $C_n \subset C_{n+1}$ for all $n \in \mathbf{N}$,

(F2) $\bigcup_{n\in\mathbf{N}} C_n = C$,

(F3) for each $n > 0$ if $C_{n-1} \neq \varnothing$ then there exists an index $i \in I$ such that for each chamber $c \in C_n$ there exists a chamber $c' \in C_{n-1}$ which is i-adjacent to c.

Lemma 3.1.: *Let $\mathcal{F}$ be a filtration. For each $n \in \mathbf{N}_0$ such that $C_{n-1} \neq \varnothing$, there exist $i \in I$ and a mapping $\pi : C_n \longrightarrow C_{n-1}$ such that for any chamber $c \in C_n$ we have $\pi(c) \sim_i c$ and $\pi_{|C_{n-1}} = \mathrm{id}_{C_{n-1}}$.*

Proof: If $c \in C_{n-1}$, then put $\pi(c) = c$. If $c \notin C_{n-1}$, then by (F3) there exists an index $i \in I$ such that there exists a chamber $c' \in C_{n-1}$ which is i-adjacent to c. We then put $\pi(c) = c'$. $\square$

A filtration $\mathcal{F} = (C_n)_{n\in\mathbf{N}}$ is called *residual* if for each $\emptyset \neq J \subset I$ and each $c \in C$ the family $(C_n \cap R_J(c))_{n\in\mathbf{N}}$ is a filtration of the chamber system $\mathcal{R}_J(c) := (R_J(c), (\sim_j)_{j\in J})$.

For each $x \in C$ we put $|x| := \min\{\lambda \in \mathbf{N} \mid x \in C_\lambda\}$. For a subset X of C we put $|X| := \min\{|x| \mid x \in X\}$ and $\mathrm{aff}(X) := \{x \in X \mid |x| = |X|\}$. Note that $C_0 = \mathrm{aff}(C)$ because we assume that $C_0 \neq \emptyset$.

Proposition 3.2.: *Let $x \in C$. Then there exist a chamber $y \in \mathrm{aff}(C)$ and a gallery $(x = x_0, \ldots, x_k = y)$ such that $|x_\lambda| < |x_{\lambda-1}|$ for all $1 \leq \lambda \leq k$.*

Proof: We will prove this by induction on $|x|$. If $|x| = 0$, then take $y = x$ and the gallery (x). Suppose the proposition is satisfied as long as $|x| < n$ and assume $|x| = n$. By (F3), there exists an index $i \in I$ such that there exists a chamber $x' \in C_{n-1}$ which is i-adjacent to x. Since $|x'| < n$, there exist a chamber $y \in \mathrm{aff}(C)$ and a gallery $(x' = x_0, \ldots, x_k = y)$ such that $|x_\lambda| < |x_{\lambda-1}|$ for all $1 \leq \lambda \leq k$. Now the chamber y and gallery $(x, x_0, \ldots, x_k = y)$ satisfy the required property since $|x_0| < |x|$. □

The following corollary is immediate.

Corollary 3.3.: *If $C_0 = \mathrm{aff}(C)$ is a connected subset of C, then C_n is a connected subset of C for all $n \in \mathbf{N}$.*

Proof: Take $x_1, x_2 \in C_n$. By the preceding proposition there exist chambers $y_1, y_2 \in \mathrm{aff}(C)$ and galleries G_1 from x_1 to y_1 and G_2 from x_2 to y_2 entirely contained in C_n. Since $\mathrm{aff}(C)$ is connected, there exists a gallery G in $\mathrm{aff}(C)$ joining y_1 to y_2. Now $G_1 G G_2^{-1}$ is a gallery from x_1 to x_2 entirely contained in C_n, hence the conclusion. □

In the remainder of this section $\mathcal{F} = (C_n)_{n \in \mathbf{N}}$ is supposed to be a residual filtration of the chamber system $\mathcal{C}$ having the property that $C_0 \neq \emptyset$.

Then the first statement of the following corollary is a consequence of the proposition above whereas the second is a consequence of the first.

Corollary 3.4.: *Let $J \subset I$, let $c \in C$ and let $R := R_J(c)$. For each $x \in R$ there exist $y \in \mathrm{aff}(R)$ and a J-gallery $(x = x_0, \ldots, x_k = y)$ such that $|x_\lambda| < |x_{\lambda-1}|$ for all $1 \leq \lambda \leq k$. If any two chambers in $\mathrm{aff}(R)$ can be connected by a J-gallery contained in $\mathrm{aff}(R)$, then any two chambers in $C_n \cap R$ can be connected by a J-gallery in $C_n \cap R$.*

Definition 3.5.: We say that the filtration $\mathcal{F}$ satisfies Condition (co) if the chamber system $(C_0, (\sim_i)_{i \in I})$ is connected.

We say that $\mathcal{F}$ satisfies Condition (lco) if for all $c \in C$ and all $J \subset I$ of cardinality at most 2, any two chambers in $\mathrm{aff}(R_J(c))$ can be joined by a J-gallery in $\mathrm{aff}(R_J(c))$. In other words: $\mathrm{aff}(R_J(c))$ is a connected subset of the chamber system $\mathcal{R}_J(c) = (R_J(c), (\sim_j)_{j \in J})$.

Definition 3.6.: Let us denote by $\mathcal{G}_j(C_n)$ $(j \in \mathbf{N})$ the set of all galleries of length j contained in C_n, and by $\mathcal{G}(C_n)$ the set of all galleries contained in C_n.

Lemma 3.7.: *Suppose the filtration $\mathcal{F}$ satisfies Condition (lco). For each $n \in \mathbf{N}_0$ such that $C_{n-1} \neq \varnothing$, there exist $i \in I$, a mapping π as in Lemma 3.1 and a mapping $\epsilon : \mathcal{G}_1(C_n) \longrightarrow \mathcal{G}(C_{n-1})$ such that:*

(E1) $\epsilon_{|\mathcal{G}_1(C_{n-1})} = \mathrm{id}_{\mathcal{G}_1(C_{n-1})}$.

(E2) $\alpha(\epsilon(x,y)) = \pi(x)$ *and* $\omega(\epsilon(x,y)) = \pi(y)$ *for all* $(x,y) \in \mathcal{G}_1(C_n)$.

(E3) If $(x,y) \in \mathcal{G}_1(C_n)$, then the chambers of $\epsilon(x,y)$, x and y are contained in a common residue of rank at most 2.

Proof: Lemma 3.1 gives us i and π. Let (x, y) be a gallery of length 1 in C_n, that is $x \sim_j y$ for $j \in I$. If x and y are in C_{n-1}, then $\pi(x) = x$, $\pi(y) = y$ and we put $\epsilon(x, y) = (x, y)$, hence (E1) is satisfied. Otherwise let R be the residue of type $\{i, j\}$ containing x. Of course R also contains, y, $\pi(x)$ and $\pi(y)$. Since (lco) holds, aff(R) is a connected subset of the chamber system $(R, (\sim_k)_{k\in\{i,j\}})$, and so, by Corollary 3.4, any two chambers in $C_{n-1} \cap R$ can be connected by a $\{i, j\}$-gallery in C_{n-1}. Let $\epsilon(x, y)$ be an $\{i, j\}$-gallery in C_{n-1} from $\pi(x)$ to $\pi(y)$. (E2) is trivially satisfied. And since $\epsilon(x, y)$ is contained in R, (E3) is also satisfied. □

Convention: For the rest of this section we suppose that the filtration $\mathcal{F}$ satisfies Condition (*lco*) and we fix a mapping $\epsilon : \mathcal{G}_1(C_n) \longrightarrow \mathcal{G}(C_{n-1})$ as in the lemma above. Furthermore we define the mapping

$$\Pi : \mathcal{G}(C_n) \longrightarrow \mathcal{G}(C_{n-1}) : G = (x_0, \dots, x_k) \mapsto \Pi(G) = (\epsilon(x_0, x_1), \epsilon(x_1, x_2), \dots, \epsilon(x_{k-1}, x_k)).$$

Proposition 3.8.: *Let $x, y \in C_0$, and let $G = (x = x_0, \dots, x_k = y)$ be a gallery from x to y which is contained in C_n for some $n > 0$. Then there exists a gallery G' from x to y which is 2-homotopic to G and which is contained in C_{n-1}.*

Proof: We will show that G is 2-homotopic to the gallery $\Pi(G)$ which is a gallery from $\pi(x) = x$ to $\pi(y) = y$ contained in C_{n-1}.

It is obvious that G is 1-homotopic to the gallery

$$\tilde{G} = (\pi(x_0), x_0, x_1, \pi(x_1), x_1, \dots, x_s, \pi(x_s), x_s, \dots, x_k, \pi(x_k)).$$

The gallery $\tilde{G}$ is elementary 2-homotopic to $(\epsilon(x_0, x_1), x_1, x_2, \pi(x_2), x_2, \dots, x_k, \pi(x_k))$, which itself is elementary 2-homotopic to $(\epsilon(x_0, x_1), \epsilon(x_1, x_2), x_2, x_3, \pi(x_3), x_3, \dots, x_k, \pi(x_k))$, and so on. One can now easily see that $\tilde{G}$ (and so G) is 2-homotopic to $\Pi(G)$.

□

We have the following two corollaries of the previous proposition.

Corollary 3.9.: *If $\mathcal{C}$ is connected, then C_n is connected for all $n \in \mathbf{N}$.*

Proof: Let $x, y \in C_0$. Since $\mathcal{C}$ is connected, there exists a gallery G from x to y in $\mathcal{C}$; this gallery is contained in C_n for some $n \geq 0$. Apply Proposition 3.8 n times to get a gallery from x to y 2-homotopic to G and contained in C_0. This proves that C_0 is connected. Now it follows by Corollary 3.3 that C_n is connected for all $n \in \mathbf{N}$. □

Corollary 3.10.: *Suppose that the chamber system $(C_0, (\sim_i)_{i\in I})$ is simply 2-connected. Then the chamber system $(C_n, (\sim_i)_{i\in I})$ is simply 2-connected for all $n \in \mathbf{N}$.*

Proof: Let G be a closed gallery in C_n with $\alpha(G) = \omega(G) = x \in C_0$. Using Proposition 3.8 several times, we get that G is 2-homotopic to a gallery G' in C_0. Since $(C_0, (\sim_i)_{i\in I})$ is simply 2-connected, G' is null-2-homotopic. Therefore G is null-2-homotopic.

Since $(C_0, (\sim_i)_{i\in I})$ is simply 2-connected, it is connected, and so is C_n by Corollary 3.3. By classical topological arguments, any closed gallery in C_n is null-2-homotopic. Hence the conclusion. □

Definition 3.11.: We say that the filtration $\mathcal{F}$ satisfies Condition (*sco*) if the chamber system $(C_0, (\sim_i)_{i\in I})$ is simply 2-connected.

We say that the filtration $\mathcal{F}$ satisfies Condition (*lsco*) if for each $c \in C$ and each $J \subset I$ of cardinality 3 the chamber system $(\text{aff}(R_J(c), (\sim_j)_{j\in J})$ is simply 2-connected.

We recall that we assume that the filtration $\mathcal{F}$ satisfies (lco).

Lemma 3.12.: *Suppose that the filtration $\mathcal{F}$ satisfies (lsco). Given two galleries G, G' contained in C_n which are elementary 3-homotopic, then there exists a 2-homotopy $(G = G_0, \ldots, G_k = G')$ such that the gallery G_λ is contained in C_n for $0 \le \lambda \le k$.*

Proof: Apply the previous corollary to the rank 3 residue in which the elementary 3-homotopy takes place. □

Lemma 3.13.: *Let $n > 0$, let G, G' be galleries in C_0 and let $(G = G_0, \ldots, G_k = G')$ be a 2-homotopy such that all galleries G_λ are contained in C_n. Then there exists a 3-homotopy $(G = G'_0, \ldots, G'_k = G')$ such that all galleries G'_λ are contained in C_{n-1}.*

Proof: Consider two successive galleries G_s and G_{s+1} in the 2-homotopy. Let R be the residue of type J with $|J| \le 2$ in which the elementary 2-homotopy takes place. The galleries $G'_s := \Pi(G_s)$ and $G'_{s+1} := \Pi(G_{s+1})$ are contained in C_{n-1}. It is easy to see that the change between these two galleries happens in the residue of type $J \cup \{i\}$ containing R, where i is as in Lemma 3.7. Hence they are elementary 3-homotopic. □

Theorem 3.14.: *Suppose that the filtration $\mathcal{F}$ satisfies (lsco). If $\mathcal{C}$ is simply 2-connected, then $(C_n, (\sim_i)_{i\in I})$ is simply 2-connected for all $n \in \mathbf{N}$.*

Proof: By Corollary 3.10, we only have to prove that $(C_0, (\sim_i)_{i\in I})$ is simply 2-connected. Let G be a closed gallery in C_0 with $\alpha(G) = \omega(G) = x$ and let $G' = (x)$. Since $\mathcal{C}$ is simply 2-connected, there exists a 2-homotopy $H = (G = G_0, G_1, \ldots, G_k = G')$ in $\mathcal{C}$. H is in C_n for some integer n. By Lemma 3.13 there exists a 3-homotopy in C_{n-1} between G and G' if $n > 0$. Using Lemma 3.12, each elementary 3-homotopy can be replaced by a 2-homotopy still in C_{n-1} and so there exists a 2-homotopy in C_{n-1} from G to G'. Using this argument n times in a row, we get a 2-homotopy in C_0 from G to G'. Hence G is null-2-homotopic in C_0. This concludes the proof. □

4 Buildings

Coxeter systems

Let G be a group. The order of an element $g \in G$ is denoted by $o(g)$.

A *Coxeter system* is a pair (W, S) consisting of a group W and a set $S \subset W$ such that $\langle S\rangle = W$, $s^2 = 1_W \ne s$ for all $s \in S$ and such that the set S and the relations $((st)^{o(st)})_{s,t\in S}$ constitute a presentation of W.

Let (W, S) be a Coxeter system. The matrix $M(S) := (o(st))_{s,t\in S}$ is called the *type* of (W, S). For an element $w \in W$ we put $l(w) := \min\{k \in \mathbf{N} \mid w = s_1 s_2 \ldots s_k$ where $s_i \in$

S for $1 \leq i \leq k\}$. The number $l(w)$ is called the *length* of w. For a subset J of S we put $W_J := \langle J \rangle$ and we call it *spherical* if W_J is finite.

In the following proposition we collect several basic facts on Coxeter groups which can be found in the usual standard references [Bo68] or [Hu90].

Proposition 4.1.: *Let (W, S) be a Coxeter system.*

a) For $w \in W, s \in S$ we have $\{l(ws), l(sw)\} \subset \{l(w) - 1, l(w) + 1\}$.

b) For $w \in W, s, t \in S$ with $l(sw) = l(w) + 1 = l(wt)$ we have $l(swt) = l(w) + 2$ or $swt = w$.

c) For $J \subset S$ the pair (W_J, J) is a Coxeter system and if $l_J : W_J \to \mathbf{N}$ is its length function, then $l_J = l \mid_{W_J}$.

d) Let $w \in W$ and $J \subset S$. Then there exists a unique element $w_J \in wW_J$ such that $l(w_J t) = l(w_J) + 1$ for all $t \in J$. Moreover, we have $l(x) = l(w_J) + l_J(w_J^{-1}x)$ for all $x \in wW_J$.

e) If $J \subset S$ is spherical, then there is a unique element $r_J \in W_J$ such that $l(r_J w) + l(w) = l(r_J)$ for all $w \in W_J$; the element r_J is a non-trivial involution if $J \neq \emptyset$.

f) For each $w \in W$ the set $J^-(w) := \{t \in S \mid l(wt) = l(w) - 1\}$ is a spherical subset of S and $l(wr_{J^-(w)}) = l(w) - l(r_{J^-(w)})$.

g) Let $w \in W$ and let $J \subset S$ be spherical. Then there exists a unique element $w^J \in wW_J$ such that $l(w^J t) = l(w^J) - 1$ for all $t \in J$ and we have $w^J = w_J r_J$. Moreover we have $l(x) = l(w^J) - l_J((w^J)^{-1}x)$ for all $x \in wW_J$; in particular, $l(w_J) + l(r_J) = l(w^J)$.

h) Let $w \in W$ and $K, J \subset S$, then there exists a unique shortest element ω in $W_K w W_J$. Moreover, if $w' \in W_K w W_J$ then $w' = \omega$ if and only if $l(w't) = l(w') + 1 = l(uw')$ for all $t \in J$ and all $u \in K$.

Buildings

Let (W, S) be a Coxeter system. A *building* of type (W, S) is a pair $\mathcal{B} = (C, \delta)$ where C is a set and where $\delta : C \times C \to W$ is a *distance function* satisfying the following axioms where $x, y \in C$ and $w = \delta(x, y)$:

(Bu 1) $w = 1$ if and only if $x = y$;

(Bu 2) if $z \in C$ is such that $\delta(y, z) = s \in S$, then $\delta(x, z) = w$ or ws, and if, furthermore, $l(ws) = l(w) + 1$, then $\delta(x, z) = ws$;

(Bu 3) if $s \in S$, there exists $z \in C$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

For a building $\mathcal{B} = (C, \delta)$ we define the chamber system $\mathbf{C}(\mathcal{B}) = (C, (\sim_s)_{s \in S})$ where two chambers $c, d \in C$ are defined to be s-adjacent if $\delta(c, d) \in \langle s \rangle$.

In this paper all buildings are assumed to be *thick* which means that for any $s \in S$ and any chamber $c \in C$ there are at least three chambers being s-adjacent to c.

For any two chambers x and y we set $l(x, y) = l(\delta(x, y))$. In the following proposition we collect several basic facts about buildings. We refer to [Ro89] and [We03] for the details.

Proposition 4.2.: *Let (W, S) be a Coxeter system and let $\mathcal{B} = (C, \delta)$ be a building of type (W, S).*

a) The chamber system $\mathbf{C}(\mathcal{B}) = (C, (\sim_s)_{s \in s})$ uniquely detemines $\mathcal{B}$; in other words, the s-adjacency relations on C determine the distance function δ.

b) For $c \in C$ and $J \subset S$ we have $R_J(c) = \{x \in C \mid \delta(c, x) \in W_J\}$.

c) If $d : C \times C \to \mathbf{N}$ is the numerical distance between two chambers in $(C, (\sim_s)_{s \in s})$, then $d = l$.

d) Let $c \in C$ and let $R \subset C$ be a J-residue for some $J \subset S$. Then there exists a unique chamber $x \in R$ such that $\delta(c, x) = (\delta(c, x))_J$. Moreover, for all $y \in R$ one has $\delta(c, y) = \delta(c, x)\delta(x, y)$ and in particular, $l(c, y) = l(c, x) + l(x, y)$.

Given $c \in C$ and a J-residue R of $\mathcal{B}$ as in Assertion d) of the previous proposition, then the chamber x of its statement is called the *projection of c onto R* and it is denoted by $\mathrm{proj}_R\, c$.

Proposition 4.3.: *Let (W, S) be a Coxeter system and let $\mathcal{B}$ be a building of type (W, S). Then the chamber system $\mathbf{C}(\mathcal{B})$ is simply 2-connected.*

Proof: This is Theorem (4.3) in [Ro89]. □

Twin buildings

Let $\mathcal{B}_+ = (C_+, \delta_+), \mathcal{B}_- = (C_-, \delta_-)$ be two buildings of the same type (W, S), where (W, S) is a Coxeter system. A *codistance* (or a *twinning*) between $\mathcal{B}_+$ and $\mathcal{B}_-$ is mapping $\delta_* : (C_+ \times C_-) \cup (C_- \times C_+) \to W$ satisfying the following axioms, where $\epsilon \in \{+, -\}, x \in C_\epsilon, y \in C_{-\epsilon}$ and $w = \delta_*(x, y)$:

(Tw 1) $\delta_*(y, x) = w^{-1}$;

(Tw 2) if $z \in C_{-\epsilon}$ is such that $\delta_{-\epsilon}(y, z) = s \in S$ and $l(ws) = l(w) - 1$, then $\delta_*(x, z) = ws$;

(Tw 3) if $s \in S$, there exists $z \in C_{-\epsilon}$ such that $\delta_{-\epsilon}(y, z) = s$ and $\delta_*(x, z) = ws$.

A *twin building of type* (W,S) is a triple $(\mathcal{B}_+,\mathcal{B}_-,\delta_*)$ where $\mathcal{B}_+,\mathcal{B}_-$ are buildings of type (W,S) and where δ_* is a twinning between $\mathcal{B}_+$ and $\mathcal{B}_-$.

For the rest of this section let $\mathcal{B}=(\mathcal{B}_+,\mathcal{B}_-,\delta_*)$ be a twin building of type (W,S), where (W,S) is a Coxeter system and where $\mathcal{B}_\epsilon=(C_\epsilon,\delta_\epsilon)$ for $\epsilon\in\{+,-\}$.

As explained in the previous subsection the buildings $\mathcal{B}_\epsilon$ can be considered as chamber systems over S. Let $J\subset S$. A J-residue in $\mathcal{B}$ is a J-residue in $\mathcal{B}_+$ or $\mathcal{B}_-$. For $x\in C_+$ and $y\in C_-$ we put $l_*(x,y)=l(\delta_*(x,y))$.

In the following proposition we collect some basic facts about twin buildings.

Proposition 4.4.: *Let $c\in C_+$, let $R\subset C_-$ be a J-residue for some $J\subset S$. Put $l_*(c,R):=\min\{l_*(c,x)\mid x\in R\}$ and $A_c(R):=\{x\in R\mid l_*(c,x)=l_*(c,R)\}$.*

a) Let $x\in R$. Then $x\in A_c(R)$ if and only if $(\delta_*(c,x))_J=\delta_*(c,x)$. Moreover, if $x,y\in A_c(R)$, then $\delta_*(c,x)=\delta_*(c,y)$.

b) Let $y\in R$. Then there exists $x\in A_c(R)$, such that $\delta_*(c,y)=\delta_*(c,x)\delta_-(x,y)$.

c) If J is spherical, then there is a unique chamber $z\in R$ such that $(\delta_*(c,z))^J=\delta_*(c,z)$. Moreover, for all $y\in R$ we have $\delta_*(c,y)=\delta_*(c,z)\delta_-(z,y)$ and in particular $l_*(c,y)=l_*(c,z)-l(z,y)$.

d) If J is a spherical subset of S, then any two J-residues of $\mathcal{B}_\epsilon$ are isomorphic for each $\epsilon\in\{+,-\}$.

Proof: Let $y\in R$ and put $w:=\delta_*(c,y)$.

Suppose first that there is $t\in J$ with $l(wt)=l(w)-1$. Choose a chamber $y'\in R$ with $y\sim_t y'\neq y$. It follows from the axioms (Tw1)-(Tw3) that $\delta_*(c,y')=wt$. Hence $y\notin A_c(R)$ in this case.

As $\{\delta_*(c,y)\mid y\in R\}\subset wW_J$ it follows from the above consideration that $A_c(R)=\{x\in R\mid \delta_*(c,x)=w_J\}$. This proves Part a) of the proposition.

Let now $t_1t_2\dots t_k$ be a reduced representation of $w_J^{-1}w$ and let $y=y_0\sim_{t_1}y_1\sim_{t_2}\dots\sim_{t_k}y_k=x$. It follows from our first consideration (by an easy induction on k) that $x\in A_c(R)$ and that $\delta_-(x,y)=w_J^{-1}w$. This finishes Part b).

Part c) follows from (4.1) in [Ro00] and Part d) is an easy consequence of (4.3) in loc. cit.. □

Let c,J and R be as in the proposition above. Then we put $\delta_*(c,R):=\delta_*(c,x)$ for some $x\in A_c(R)$. It follows from Parts a) and b) of the proposition that $\delta_*(c,R)$ is well-defined and that $l(\delta_*(c,R))=l_*(c,R)$.

Let c,J and R be as in Assertion c) of the previous proposition, i.e. we suppose now in addition that J is a spherical subset of S. As before, the chamber x in its statement is called the *projection* of c onto R and denoted by $\mathrm{proj}_R\,c$. Moreover, the following observation is an easy consequence of the previous proposition.

Corollary 4.5.: *Let c,J,R be as above and suppose that J is spherical. Then $A_c(R)=\{x\in R\mid \delta_-(\mathrm{proj}_R\,c,x)=r_J\}$.*

The following proposition is the 'twin version' of the main result in [DS87]. The proof given there generalizes without much change to the situation of twin buildings.

Proposition 4.6.: *Let R, Q be two spherical residues of a twin building. Put* $\operatorname{proj}_R Q := \{\operatorname{proj}_R x \mid x \in Q\}$. *Then the following holds:*

a) $\operatorname{proj}_R Q$ *is a spherical residue contained in* R.

b) If $R' := \operatorname{proj}_R Q$ *and* $Q' := \operatorname{proj}_Q R$, *then* $\operatorname{proj}_{R'}^{Q'} := \operatorname{proj}_{R'} |_{Q'}\colon Q' \to R'$ *and* $\operatorname{proj}_{Q'}^{R'} := \operatorname{proj}_{Q'} |_{R'}\colon R' \to Q'$ *are adjacence-preserving bijections inverse to each other.*

5 The chamber system far away from a chamber

Throughout this section let $\Delta = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ be a twin building of type (W, S) where (W, S) is a Coxeter system and where $\mathcal{B}_\epsilon = (C_\epsilon, \delta_\epsilon)$ for $\epsilon \in \{+, -\}$.

The filtration $\mathcal{F}_c$

We choose an injection $w \mapsto |w|$ from W onto $\mathbf{N}$ such that $l(x) < l(y)$ implies $|x| < |y|$ for all $x, y \in W$ and such that $|1_W| = 0$. Moreover, we fix a chamber $c \in C_+$. We define C_n by setting $C_n := \{x \in C_- \mid |\delta_*(c, x)| \leq n\}$. The goal of this subsection is to show the following proposition.

Proposition 5.1.: *With the definitions above, the family* $\mathcal{F}_c := (C_n)_{n \in \mathbf{N}}$ *is a residual filtration of the chamber system* $\mathbf{C}(\mathcal{B}_-)$.

Proof: It is obvious that $\mathcal{F}_c$ satisfies the axioms (F1) and (F2) and from this it follows that these axioms also hold 'residually'.

Let R be a J-residue of $\mathbf{C}(\mathcal{B}_-)$ with $J \neq \emptyset$. and let $|R| := \min\{k \mid C_k \cap R \neq \emptyset\}$. It follows from the definition of $\mathcal{F}_c$ and by Proposition 4.4 that $C_{|R|} \cap R = A_c(R) = \{x \in R \mid \delta_*(c, x) = \delta_*(c, R)\}$ and that $|\delta_*(c, R)| = |R|$.

Let $0 < n \in \mathbf{N}$ be such that $C_{n-1} \cap R \neq \emptyset$. We have to show that there is $t \in J$ with the property that each chamber x in $R \cap C_n$ is t-adjacent to a chamber $x' \in R \cap C_{n-1}$. If $C_n \cap R = C_{n-1} \cap R$ we can choose $t \in J$ arbitrarily and set $x' := x$ for each $x \in R \cap C_n$. Suppose now that $R \cap C_{n-1}$ is properly contained in $C_n \cap R$, choose $y \in C_n \setminus C_{n-1}$ and put $w := \delta_*(c, y)$. Since $||$ injects W into $\mathbf{N}$, it follows from the definition of $\mathcal{F}_c$ that $\delta_*(c, y') = w$ for all $y' \in C_n \setminus C_{n-1}$. On the other hand, there exists $x \in A_c(R)$ such that $w = \delta_*(c, y) = \delta_*(c, x)\delta_-(x, y)$ by Assertion b) of Proposition 4.4. As $C_{n-1} \cap R \neq \emptyset$ it follows that $y \notin A_c(R)$ and hence $\delta_-(x, y) \in W_J \setminus \{1_W\}$. Let $t \in J$ be such that $l(\delta_-(x, y)t) = l(\delta_-(x, y)) - 1$. As $\delta_*(c, x) = (\delta_*(c, x))_J$ and $\delta_-(x, y) \in W_J$ (by Part b) of Proposition 4.2) it follows that $l(wt) = l(w) - 1$ by Part d) of Proposition 4.1. For any chamber $z \in R \cap C_n$ we choose a chamber $z' \in R$ as follows. If $z \in C_{n-1}$ then we put $z' := z$. If $z \in C_n \setminus C_{n-1}$ then we choose $z' \in R$ such that $z \sim_t z' \neq z$. In the first case, it is obvious that z' is in $R \cap C_{n-1}$; in the second case we have $\delta_*(c,z) = w$ and as $l(wt) = l(w) - 1$ it is a direct consequence of the axioms for a twinning that $\delta_*(c,z') = wt$. As $l(wt) = l(w) - 1$ it follows that $|wt| < |w| = n$ and therefore $z' \in C_{n-1}$. As $t \in J$ we have also $z' \in R$.

The case $J := S$ is a special case of the consideration above. This shows that $\mathcal{F}_c$ satisfies Axiom (F3). Hence $\mathcal{F}_c$ is a residual filtration. □

A criterion for the simple connectedness of c^{op}

For $c \in C_+$ we put $c^{\mathrm{op}} := \{x \in C_- \mid \delta_*(c,x) = 1_W\}$. The chamber system $(c^{\mathrm{op}}, (\sim_s)_{s \in S})$ is called the chamber system *far away from c*. Here is the main result of this section.

Theorem 5.2.: *Suppose that the following conditions are satisfied:*

a) If $J \subset S$ is of cardinality at most 3, then J is spherical.

b) If J is of cardinality 2, if $R \subset C_-$ is a J-residue and if $x \in R$, then the chamber system $(\{y \in R \mid \delta_-(x,y) = r_J\}, (\sim_t)_{t \in J})$ is connected.

c) If J is of cardinality 3, if $R \subset C_-$ is a J-residue and if $x \in R$, then the chamber system $(\{y \in R \mid \delta_-(x,y) = r_J\}, (\sim_t)_{t \in J})$ is simply 2-connected.

Then the chamber system far away from a chamber $c \in C_+$ is simply 2-connected.

Proof: Let $c \in C_+$ and let $\mathcal{F}_c = (C_n)_{n \in \mathbf{N}}$ be the residual filtration of the previous subsection. Note first that $C_0 = c^{\mathrm{op}}$.

For a set $X \subset C_-$ define $\mathrm{aff}_c(X)$ as in Section 3 with respect to the filtration $\mathcal{F}_c$.

Given a spherical J-residue R of $\mathcal{B}_-$, then $\mathrm{aff}_c(R) = A_c(R)$ and by Corollary 4.5 we have therefore $\mathrm{aff}_c(R) = \{x \in R \mid \delta_-(\mathrm{proj}_R\, c, x) = r_J\}$ for each such residue.

Now $\mathcal{F}_c$ satisfies (lco) by Assumption b) and it satisfies (lsco) by Assumption c). As $\mathbf{C}(\mathcal{B}_-)$ is simply 2-connected by Proposition 4.3 the claim follows now from Theorem 3.14. □

Proof of Theorem 1.1:

Theorem 1.1 follows from Theorem 5.2 and Part d) of 4.4.

6 Flipflop chamber systems

Involutions and flips of twin buildings

Lemma 6.1.: *Let (W,S) be a Coxeter system, let $w \in W$ be an involution and let $s \in S$ be such that $l(ws) = l(w) - 1$. Then $l(sws) = l(w) - 2$ or $ws = sw$. In particular, ws or sws are involutions of strictly smaller length than w.*

Proof: Let $s_1 \dots s_k$ be a reduced representations of ws. Then $s_1 \dots s_k s$ and $ss_k \dots s_1$ are reduced representation of w and hence $l(sw) = l(w) - 1$. If $l(sws) \neq l(w) - 2$, then $l(sws) \neq l(sw) - 1$ and therefore $l(sws) = l(sw) + 1 = l(w) = l(ws) + 1$ Now $l(s(ws)) = l(w) = l((ws)s)$ and as $l(w) = l(ws) + 1$ we have that s commutes with ws by the Exchange property (Part b) of Proposition 4.1) and therefore s commutes with w. $\square$

Definition 6.2.: Let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ be a twin building of type (W, S) where (W, S) is a Coxeter system and where $\mathcal{B}_\epsilon = (C_\epsilon, \delta_\epsilon)$ for $\epsilon \in \{+, -\}$. An involution of $\mathcal{B}$ is an involutory permutation τ of $C_+ \cup C_-$ such that the following holds for all $\epsilon \in \{+, -\}$.

(In1) $\tau(C_\epsilon) = C_{-\epsilon}$;

(In2) $\delta_{-\epsilon}(\tau(x), \tau(y)) = \delta_\epsilon(x, y)$ for all $x, y \in C_\epsilon$;

(In3) $\delta_*(\tau(x), \tau(y)) = \delta_*(x, y)$ for all $x \in C_\epsilon, y \in C_{-\epsilon}$.

Let τ be an involution of $\mathcal{B}$. For each spherical residue R of $\mathcal{B}$ we put

$$\operatorname{proj}_R \tau := \{x \in R \mid \operatorname{proj}_R \tau(x) = x\}.$$

An involution τ of $\mathcal{B}$ is called a flip if $\operatorname{proj}_P \tau \neq P$ for each panel of $\mathcal{B}$.

Remark: Flips of twin buildings have been defined in [BGHS] in a different way. It is not hard to see that an involution satisfying our conditions is also a flip in the sense loc.cit.. However, we do not know whether the definitions are equivalent. Anyhow, our definition of a flip covers the cases in which one is interested, namely those where the involution in question is twisted with an involutory field automorphism. We prefered to use the definition above because it is much more convenient for our purposes in the sense that we can avoid several technical details.

The following lemma is immediate.

Lemma 6.3.: *Let c be a chamber of a twin building $\mathcal{B}$ and let τ be an involution of $\mathcal{B}$. Then $w := \delta_*(c, \tau(c))$ is an involution. Moreover, if $s \in S$ and if P is the s-panel containing c then $c \in \operatorname{proj}_P \tau$ if and only if $l(ws) = l(w) - 1$.*

Proposition 6.4.: *Let $\mathcal{B}$ be a twin building, let τ be a flip of $\mathcal{B}$ and let R be a J-residue of $\mathcal{B}$. Put $l_*(\tau, R) := \min\{l(\delta_*(x, \tau(x)) \mid x \in R\}$ and $A_\tau(R) := \{x \in R \mid l(\delta_*(x, \tau(x)) = l_*(\tau, R)\}$. Then the following holds:*

a) For $x, y \in A_\tau(R)$ one has $\delta_*(x, \tau(x)) = \delta_*(y, \tau(y))$.

b) If $x \in R$ is such that $l(\delta_*(x, \tau(x))t) = l(\delta_*(x, \tau(x)) + 1$ for all $t \in J$, then $x \in A_\tau(R)$.

Proof: Let $x \in R$ and let $w := \delta_*(x, \tau(x))$.

Suppose first that $l(wt) = l(w) - 1$ for some $t \in J$. If P denotes the t-panel containing x, it follows that $x \in \operatorname{proj}_P \tau$. As τ is supposed to be a flip there exists a chamber $y \in P$ with $y \notin \operatorname{proj}_P \tau$ which implies that $l(\delta_*(y, \tau(y))) < l(w)$ by Lemma 6.1. Hence $x \notin A_\tau(R)$.

The sets $Y := \{\delta_*(x, \tau(x)) \mid x \in R\}$ and $W_J w W_J$ are equal and by Part h) of Proposition 4.1 there exists a unique shortest element in Y which we denote by ω. It follows from the previous consideration that $A_\tau(R) = \{x \in R \mid \delta_*(x, \tau(x)) = \omega\}$ which yields Assertion a).

Suppose now that $l(wt) = l(w) + 1$ for all $t \in J$. As w is an involution it follows that $l(tw) = l(w) + 1$ for all $t \in J$ and by by the second assertion of Part h) of Proposition 4.1 we conclude that $w = \omega$, and therefore $x \in A_\tau(R)$.

□

The filtration $\mathcal{F}_\tau$

Throughout this section let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ be a twin building of type (W, S) where (W, S) is a Coxeter system and where $\mathcal{B}_\epsilon = (C_\epsilon, \delta_\epsilon)$ for $\epsilon \in \{+, -\}$ and let τ be a flip of $\mathcal{B}$. For each residue R define the set $A_\tau(R)$ as in Proposition 6.4.

Let $\mathrm{Inv}(W) := \{w \in W \mid w^2 = 1_W\}$ and let $x \mapsto |x|$ be an injective mapping from $\mathrm{Inv}(W)$ into $\mathbf{N}$ such that $|1_W| = 0$ and such that $l(x) < l(y)$ implies $|x| < |y|$ for all $x, y \in \mathrm{Inv}(W)$. Define $C_n := \{c \in C_+ \mid |\delta_*(c, \tau(c))| \leq n\}$. We have the following proposition.

Proposition 6.5.: *With the definitions above, the family $\mathcal{F}_\tau := (C_n)_{n \in \mathbf{N}}$ is a residual filtration of $\mathbf{C}(\mathcal{B}_+)$.*

Proof: It is clear from the definitions that $\mathcal{F}_\tau$ satisfies axioms (F1) and (F2) and that these axioms are also satisfied the families $(R \cap C_n)_{n \in \mathbf{N}}$ for each residue R of $\mathcal{B}_+$.

Let $\emptyset \neq J \subset S$, let R be a J-residue of $\mathbf{C}(\mathcal{B}_+)$ and put $|R| := \min\{k \in \mathbf{N} \mid C_k \cap R \neq \emptyset\}$. It follows from Proposition 6.4 Part a) and from the definition of $\mathcal{F}_\tau$ that $C_{|R|} \cap R = A_\tau(R)$.

Let $0 < n \in \mathbf{N}$ be such that $C_{n-1} \cap R \neq \emptyset$. We have to show that there is $t \in J$ with the property that each chamber x in $R \cap C_n$ is t-adjacent to a chamber $x' \in R \cap C_{n-1}$. If $C_n \cap R = C_{n-1} \cap R$ we can choose $t \in J$ arbitrarily and set $x' := x$ for each $x \in R \cap C_n$. Suppose now that $R \cap C_{n-1}$ is properly contained in $C_n \cap R$, choose $y \in C_n \setminus C_{n-1}$ and put $w := \delta_*(y, \tau(y))$. It follows from the definition of $\mathcal{F}_\tau$ that $\delta_*(y', \tau(y')) = w$ for all $y' \in C_n \setminus C_{n-1}$. As $C_{n-1} \cap R \neq \emptyset$ it follows that $y \notin A_\tau(R)$ and hence there exists $t \in J$ such that $l(wt) = l(w) - 1$ by Part c) of Proposition 6.4 and Part a) of Proposition 4.1. For any chamber $z \in R \cap C_n$ we choose a chamber $z' \in R$ as follows. If $z \in C_{n-1}$ then we put $z' := z$. If $z \in C_n \setminus C_{n-1}$ then, by Lemma 6.3, $\mathrm{proj}_P \tau(z) = z$ where P is the t-residue containing z. By Proposition 4.6, two situations can occur: either $\mathrm{proj}_P \tau(P)$ is P itself, or $\mathrm{proj}_P \tau(P)$ is a single point, which can only be z by the remark above.

In the first case, we choose z' in P (so that z' is t-adjacent to z) such that $\mathrm{proj}_P z' \neq z'$ (which is possible because τ is supposed to be a flip), hence $\delta_*(z', \tau(z')) = wt$. Since $l(wt) = l(w) - 1$, $z' \in C_{n-1}$, and $z' \in R$ since $t \in J$.

In the second case, we choose $z' \neq z$ in P. Then $\delta_*(z', \tau(z')) = twt$, $l(twt) = l(w) - 2$, and we reach the same conclusion.

The case $J := S$ is a special case of the above consideration which shows that $\mathcal{F}_c$ satisfies Axiom (F3). Hence $\mathcal{F}_c$ is a residual filtration. □

A criterion for the simple connectedness of τ^{op}

Let τ be a flip of the twin building $\mathcal{B}$. We put $\tau^{\mathrm{op}} := \{c \in C_+ \mid \delta_*(c, \tau(c)) = 1_W\}$. Here is the main result of this section.

Theorem 6.6.: *Suppose that the following conditions are satisfied:*

a) If $J \subset S$ is of cardinality at most 3, then J is spherical.

b) If $J \subset S$ is of cardinality 2, if $R \subset C_+$ is a J-residue then the chamber system $(A_\tau(R), (\sim_t)_{t\in J})$ is connected.

c) If $J \subset S$ is of cardinality 3, if $R \subset C_+$ is a J-residue then the chamber system $(A_\tau(R), (\sim_t)_{t\in J})$ is simply 2-connected.

Then the chamber system $(\tau^{\mathrm{op}}, (\sim_s)_{s\in S})$ is simply 2-connected.

Proof: Let $\mathcal{F}_\tau = (C_n)_{n\in\mathbf{N}}$ be the residual filtration of the previous subsection. Note first that $C_0 = \tau^{\mathrm{op}}$.

For a set $X \subset C_+$ define $\mathrm{aff}_\tau(X)$ as in Section 3 with respect to the filtration $\mathcal{F}_\tau$.

Given a spherical J-residue R of $\mathcal{B}_+$, then $\mathrm{aff}_\tau(R) = A_\tau(R)$.

Now $\mathcal{F}_\tau$ satisfies (lco) by Assumption b) and it satisfies (lsco) by Assumption c). By Proposition 4.3 the chamber system $\mathbf{C}(\mathcal{B}_+)$ is simply 2-connected and therefore the claim follows now from Theorem 3.14. □

References

[Ab95] P. Abramenko: *Twin buildings and applications to S-arithmetic groups.* Lecture Notes in Mathematics, **1641**, Springer-Verlag, Berlin, 1996.

[Ab02] P. Abramenko: *Finiteness properties of groups acting on twin buildings.* in *Groups: topological, combinatorial and arithmetic aspects*, London Math. Soc. Lecture Note Ser., **311**, Cambridge Univ. Press, Cambridge, 2004, 21–26.

[AM97] P. Abramenko, B. Mühlherr: *Présentation des certaines BN-paires jumelées comme sommes amalgamées.* C. R. Acad. Sci. Paris, t. 325, Série I, p. 701–706, 1997.

[BGHS] C. Bennett, R. Gramlich, C. Hoffmann, S. Shpectorov: *Curtis-Phan-Tits theory.* in *Groups, combinatorics and geometry (Durham, 2001)*, World Sci. Publishing, River Edge, NJ, 2003, 13–29.

[Bo68] N. Bourbaki: *Groupes et Algèbres de Lie, Chapitres 4,5 et 6.* Hermann, Paris, 1968.

[Br93] A. E. Brouwer: *The complement of a geometric hyperplane in a generalized polygon is usually connected.*, in *Finite Geometry and Combinatorics.*, ed F. De Clerck et al., Cambridge University Press, 1993.

[DS87] A. Dress, R. Scharlau: *Gated sets in metric spaces.* Aequationes Math. **34** (1987), 112–120.

[GHMS] R. Gramlich, C. Hofmann, B. Mühlherr, S. Shpectorov: *Phan presentations of Chevalley groups and Kac-Moody-groups*, in preparation.

[Hu90] J.E. Humphreys: *Reflection groups and Coxeter groups.* Cambridge Studies in Advanced Mathematics, **29**, Cambridge University Press, Cambridge, 1990

[MR95] B. Mühlherr, M. Ronan: *Local to Global Structure in Twin Buildings.* Invent. math. **122** (1995), 71–81.

[Ph77] K.-W. Phan: *On groups generated by three-dimensional special unitary groups. I.,II* J. Austral. Math. Soc. Ser. A 23 (1977), no. 1, 67–77, no. 2, 129–146.

[Ro89] M. Ronan: *Lectures on Buildings.* Academic Press, San Diego, 1989.

[Ro00] M. Ronan: *Local isometries of twin buildings.* Math. Z. **234** (2000), 435–455.

[Ti81] *A local approach to buildings.* The geometric vein, Springer, New York-Berlin, 1981, pp 519–547.

[Ti86] J. Tits: *Ensembles ordonnés, immeubles et sommes amalgamées.* Bull. Soc. Math. Belg. Sér. **A 38** (1986), 367–387.

[Ti88/89] J. Tits: *Résumé de cours.* Annuaire du Collège de France, 89e année, 1988–1989,81–95.

[Ti89/90] J. Tits: *Résumé de cours.* Annuaire du Collège de France, 90e année, 1989–1990,87–103.

[Ti90] J. Tits: *Twin Buildings and groups of Kac-Moody type.* In M.W.Liebeck and J. Saxl(eds.), *Groups, Combinatorics and Geometry* (Durham 1990), London Math. Soc. Lecture Note Ser. **165**, Cambridge University Press, 249–286.

[We03] R. Weiss: *The structure of spherical buildings.* Princeton University Press, Princeton, 2003.