Gain and phase margin

Two-mass problem: (non-colocated)

At neutral stability,

\[ |gG(j\omega)H(j\omega)| = 1 \]
\[ \phi(gGH) = -\pi \]

Stability condition:

\[ \phi > -\pi \quad \text{whenever} \quad |gGH| > 1 \]

Root locus:

\[ 1 + gG(s)H(s) = 0 \]
\[ |gG(s)H(s)| = 1 \quad \phi(gGH) = -\pi \]
If $G(s)$ has no singularity within $C_1$, $C_2$ does not encircle the origin.
If there is 1 zero within $C_1$, $C_2$ has 1 encirclement clockwise.
If ... 1 pole ... clockwise.

Number of clockwise encirclements: $N = Z - P$
Nyquist criterion

Number of instable poles of 

\[ F(s) = \frac{G(s)}{1 + G(s)} \]

Contour C1 encircling the right half plane:

If \( G(s) \) has poles on the imaginary axis
If $G'(s) = \frac{n(s)}{d(s)}$, the closed-loop poles are solution of:

$$1 + G'(s) = \frac{d(s) + n(s)}{d(s)} = 0$$

- The poles of $1 + G(s)$ are the same as those of $G(s)$
- The zeros of $1 + G(s)$ are the closed-loop poles of $F(s)$

Unstable poles of $F(s)$

Encirclements of $-1$ by $G(s)$

Unstable poles of $G(s)$

Mapping $1 + G(s)$:

\[ Z = N + P \]

\[ N = Z - P \]
Gain and phase margin in the Nyquist plot

\[ G(j\omega) \]

Unit circle

\[ \frac{\sqrt{GM}}{\omega_c} \]

PM
Non-colocated Two-mass system
F(s) and G(s) are uniquely related. The loci of constant magnitude $|F| = M$ are circles.

$F(s)$ and $G(s)$ are uniquely related. The loci of constant magnitude $|F| = M$ are circles.

Since: $\omega_b \sim \omega_p \sim \omega_c$

Direct relation between PM and $\omega_p$

No overshoot if PM > 60°
Nichols chart

- Amplifies the vicinity of -1
- Allows a summation of the contributions of G and H
\(\xi = 0.02\)

- Neutrally stable

\(\leftarrow\) GM = \(\infty\)

PM \approx 70^\circ
1) Sensitivity

\[ \frac{y}{r} = \frac{G}{1 + G} \]

\[ \frac{e}{r} = \frac{1}{1 + G} \]

\[ \frac{\delta F}{F} = \frac{1}{1 + G} \frac{\delta G}{G} \]

In the frequency range where \(|1 + G| \gg 1\), the sensitivity of \(F\) to the change of parameters is much smaller than that of \(G\).
2) Performance specification

\[ e = r - y - n \]
\[ y = Ge + d = G(r - y - n) + d \]
\[ y = \frac{G}{1 + G}(r - n) + \frac{1}{1 + G}d \]

Tracking error:
\[ e^* = r - y = \frac{1}{1 + G}(r - d) + \frac{G}{1 + G}n \]

Performance:
\[ |1 + GH(\omega)| \geq ps(\omega) \]
\[ \omega \leq \omega_0 \]

Stability !!

Noise rejection:
\[ |GH(\omega)| \ll 1 \quad \omega > \omega_1 \]
3) Unstructured uncertainty

Additive:
\[ G' (\omega) = G(\omega) + \Delta G(\omega), \quad |\Delta G(\omega)| < l_\alpha (\omega), \quad \omega > 0 \]

Multiplicative:
\[ G'(\omega) = G(\omega) [1 + L(\omega)], \quad |L(\omega)| < l_m(\omega), \quad \omega > 0 \]
4) Stability in face of uncertainty

\[
\frac{1}{l_m(\omega)} \geq \left| \frac{GH}{1 + GH} \right|
\]
5) Performance robustness

\[ |1 + (1 + L)GH| \geq ps(\omega) \quad \omega \leq \omega_0 \]

Perturbed system

\[ |GH| \geq \frac{ps(\omega)}{1 - l_m(\omega)} \]

For stability, the phase should remain > -180° whenever |GH| > 1
6) Behaviour near crossover, $|GH|=1$

**At Crossover:**

$$|1 + GH| = 2 \sin\left(\frac{PM}{2}\right)$$

There is a direct relation between $PM$ and the disturbance rejection near crossover.

The stability robustness requires that:

$$|1 + GH| \geq l_m(\omega_c)$$

Accepting a magnitude error $l_m(\omega_c) = 1$ at crossover requires $PM = 60^\circ$.
7) BODE gain-phase relationships

Constant slope of n poles \([n \times (-20 \text{ dB/decade})]\): \(\text{phase}= - n \times 90^\circ\)

**General case**

For a minimum phase system, the phase angle and the amplitude ar uniquely related

\[
\phi(\omega_0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\ln |G|}{du} W(u) du
\]

\((u = \ln(\omega/\omega_0))\)

where \(W(u) = \ln[\coth(|u|/2)]\)

A large phase can only be achieved if the gain attenuates slowly
Note:

\[ W(u) = \ln[ \coth(|u|/2)] \approx \frac{1}{2} \pi^2 \delta(u) \]

\[ \phi(\omega_0) \approx \frac{\pi}{2} \frac{d \ln |G|}{du} \bigg|_{u=0} \] (this relation applies if the slope is nearly constant near \( u=0 \))

-20 dB/decade \( \Rightarrow \frac{d \ln |G|}{du} = -1 \) \( \Rightarrow \phi = -\pi/2 \)

-40 dB/decade \( \Rightarrow \frac{d \ln |G|}{du} = -2 \) \( \Rightarrow \phi = -\pi \)

A slow attenuation rate will be required near crossover to achieve PM
For a stable $G(s)$ with more than one pole roll-off ($n > 1$):

\[ \int_0^\infty \ln |1 + G| \, d\omega = 0 \]

If $|1 + G| > 1$: negative feedback
$|1 + G| < 1$: positive feedback

Negative feedback is always balanced by positive feedback

Good disturbance rejection in some frequency range can only be achieved by making things worse outside that frequency band!
Integral # 3

Reshaping of $|G|$ can be done within the working band without affecting the phase outside the working band.

\[
\int_{\omega=0}^{\omega=1} \left( \ln |G| - \ln |G|_{\infty} \right) d \arcsin \omega = - \int_{1}^{\infty} \frac{\phi}{\sqrt{\omega^2 - 1}} d\omega
\]
The greater the phase lag, the greater will be the feedback in the working band:

$$\int_{-\infty}^{\infty} \frac{\phi}{\omega} d\omega = \pi \{ \ln |G|_{\infty} - \ln |G|_0 \}$$

In particular, if $G_1$ and $G_2$ have the same high frequency behaviour, but $G_2$ has a greater phase lag than $G_1$, $G_2$ has a larger magnitude than $G_1$ in the working band:

$$\int_{-\infty}^{\infty} \frac{\phi_1 - \phi_2}{\omega} d\omega = \pi \ln \left| \frac{G_2}{G_1} \right|_0$$
8) BODE Ideal Cutoff

- Requested $GM = x \text{ dB}$
- $PM = y \pi$

Constant feedback as large as possible within the working band

Constant phase beyond the working band

Nichols chart of the two segment problem
Bode plot of the Ideal cutoff (two segment problem)

Curves for two values of PM

PM1 = 90°
PM2 = 60°

\[ G(j\omega) = \frac{|G|_0}{\sqrt{1 - \omega^2 + j\omega}}^{\frac{2\phi}{\pi}} \]
If a higher roll-off is needed at higher frequency (for sensor noise attenuation)  
Need to include a flat segment to attenuate its effect on PM
Design tradeoff

Robust performance
Tracking error & disturbance rejection

\[ \frac{ps(\omega)}{1-l_m(\omega)} \]

Stability robustness
Sensor noise rejection

\[ \frac{GH}{1+GH} < \sqrt{I_m} \]

Sensor noise

\[ \omega_c \]

BODE Ideal cutoff

|G|dB

-40(1-y) dB/ decade

\[ \omega_c \]

1-2 octaves

-20 dB/ decade

\[ \phi = (1-y)\pi \]

Working band