# Partial order production problem

"INFO-F-413"

Keno Merckx

Université Libre de Bruxelles

13 décembre 2012

# Definition (1)

First, we need some basic definitions:

**Definition (poset)**

A partially ordered set is a pair $P = (X, \leq)$ with $X$ a set and $\leq$ an antisymmetric, reflexive and transitive relation on $X$.

# Definition (1)

First, we need some basic definitions:

## Definition (poset)

A partially ordered set is a pair $P = (X, \leq)$ with $X$ a set and $\leq$ an antisymmetric, reflexive and transitive relation on $X$.

## Definition (covering relation)

The covering relation of a poset $P$, written $\lhd$, is defined as:

$$x \lhd y \Leftrightarrow \begin{cases} x \leq y \\ \exists w : x \leq w \leq y \Rightarrow w = x \text{ or } w = y. \end{cases}$$

# Definition (1)

First, we need some basic definitions:

## Definition (poset)

A partially ordered set is a pair $P = (X, \leq)$ with $X$ a set and $\leq$ an antisymmetric, reflexive and transitive relation on $X$.

## Definition (covering relation)

The covering relation of a poset $P$, written $\lhd$, is defined as:

$$x \lhd y \Leftrightarrow \begin{cases} x \leq y \\ \exists w : x \leq w \leq y \Rightarrow w = x \text{ or } w = y. \end{cases}$$

## Examples

If $E_n = \{0, 1, \ldots, n\}$,

- then $(E_n, \leq_{\mathbb{R}})$ is a poset. And $i \lhd i + 1$, $(\forall i \in E_{n-1})$.
- then $(\mathcal{P}(E_n), \subseteq)$ is a poset. And $I \lhd J$ iff $I \subsetneq J$ and $|I| + 1 = |J|$.

# Definition (2)

**Definition (Hasse diagram)**

The Hasse diagram of a poset $P$ is a drawing of $P$ in the plane such that:

- Each $x \in X$ is represented by a point $p_x$,
- There is a straight line that goes upward from $p_x$ to $p_y$ whenever $x \lhd y$.

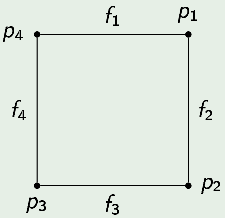
# Definition (2)

## Definition (Hasse diagram)

The Hasse diagram of a poset $P$ is a drawing of $P$ in the plane such that:

- Each $x \in X$ is represented by a point $p_x$,
- There is a straight line that goes upward from $p_x$ to $p_y$ whenever $x \lhd y$.

## Example: Hasse($\mathcal{F}(\square), \subseteq$)



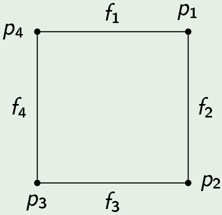
# Definition (2)

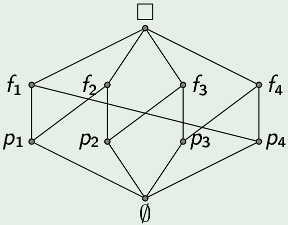
## Definition (Hasse diagram)

The Hasse diagram of a poset $P$ is a drawing of $P$ in the plane such that:

- Each $x \in X$ is represented by a point $p_x$,
- There is a straight line that goes upward from $p_x$ to $p_y$ whenever $x \lhd y$.

## Example: Hasse($\mathcal{F}(\square), \subseteq$)



# P.O.P.

We consider the following problem:

## Partial Order Production problem

Given a set $S = \{s_1, s_2, \ldots, s_n\}$ partially ordered by a known partial order $\preceq$ and a set $T = \{t_1, t_2, \ldots, t_n\}$ totally ordered by an unknown linear order $\leq$, find a permutation $\pi$ of $\{1, 2, \ldots, n\}$ such that $s_i \preceq s_j \Rightarrow t_{\pi(i)} \leq t_{\pi(j)}$, by asking questions of the form: "is $t_i \leq t_j$ ?".

# P.O.P.

We consider the following problem:

**Partial Order Production problem**

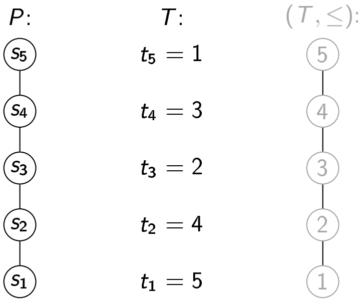
Given a set $S = \{s_1, s_2, \ldots, s_n\}$ partially ordered by a known partial order $\preceq$ and a set $T = \{t_1, t_2, \ldots, t_n\}$ totally ordered by an unknown linear order $\leq$, find a permutation $\pi$ of $\{1, 2, \ldots, n\}$ such that $s_i \preceq s_j \Rightarrow t_{\pi(i)} \leq t_{\pi(j)}$, by asking questions of the form: "is $t_i \leq t_j$ ?".

This generalizes many fundamental problems:

- sorting by comparisons,
- (multiple) selection problem,
- heap construction.

# Special case: sorting

If $P := (S, \preceq)$ is a chain.

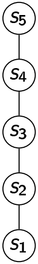



We must find $\pi$

# Special case: sorting

If $P := (S, \preceq)$ is a chain.

$P$:




$T$:

$t_5 = 1$

$t_4 = 3$

$t_3 = 2$

$t_2 = 4$

$t_1 = 5$

$(T, \leq)$:


We must find $\pi$

$\pi = (1,5)(2,3,4)$

$t_{\pi(5)} = 5$

$t_{\pi(4)} = 4$

$t_{\pi(3)} = 3$

$t_{\pi(2)} = 2$

$t_{\pi(1)} = 1$

## Special case: sorting

If $P := (S, \preceq)$ is a chain.

| $P$: | $T$: | $(T, \leq)$: | | $\pi = (1,5)(2,3,4)$ |
|---|---|---|---|---|
| $s_5$ | $t_5 = 1$ | 5 | | $t_{\pi(5)} = 5$ |
| $s_4$ | $t_4 = 3$ | 4 | We must | $t_{\pi(4)} = 4$ |
| $s_3$ | $t_3 = 2$ | 3 | find $\pi$ | $t_{\pi(3)} = 3$ |
| $s_2$ | $t_2 = 4$ | 2 | | $t_{\pi(2)} = 2$ |
| $s_1$ | $t_1 = 5$ | 1 | | $t_{\pi(1)} = 1$ |

# Special case: sorting

If $P := (S, \preceq)$ is a chain.

$P$:


$T$:

$t_5 = 1$

$t_4 = 3$

$t_3 = 2$

$t_2 = 4$

$t_1 = 5$

$(T, \leq)$:


We must find $\pi$

$\pi = (1,5)(2,3,4)$

$t_{\pi(5)} = 5$

$t_{\pi(4)} = 4$

$t_{\pi(3)} = 3$

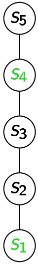
$t_{\pi(2)} = 2$

$t_{\pi(1)} = 1$

# Special case: sorting

If $P := (S, \preceq)$ is a chain.

$P$:




$T$:

$t_5 = 1$

$t_4 = 3$

$t_3 = 2$

$t_2 = 4$

$t_1 = 5$

$(T, \leq)$:


We must find $\pi$

$\pi = (1,5)(2,3,4)$

$t_{\pi(5)} = 5$

$t_{\pi(4)} = 4$

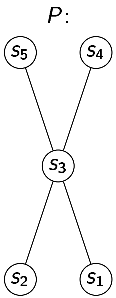
$t_{\pi(3)} = 3$

$t_{\pi(2)} = 2$

$t_{\pi(1)} = 1$

# Special case: selection

If $P := (S, \preceq)$ is a particular weak order.




$T$:

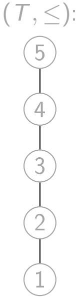
$t_5 = 3$

$t_4 = 5$

$t_3 = 4$

$t_2 = 2$

$t_1 = 1$

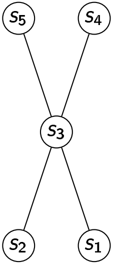



We must find $\pi$

# Special case: selection

If $P := (S, \preceq)$ is a particular weak order.

$P$:




$T$:

$t_5 = 3$

$t_4 = 5$

$t_3 = 4$

$t_2 = 2$

$t_1 = 1$

$(T, \leq)$:


We must find $\pi$

$\pi = (3, 5)$

$t_{\pi(5)} = 4$

$t_{\pi(4)} = 5$

$t_{\pi(3)} = 3$

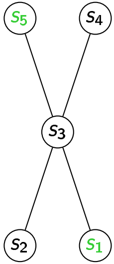
$t_{\pi(2)} = 2$

$t_{\pi(1)} = 1$

# Special case: selection

If $P := (S, \preceq)$ is a particular weak order.

$P$:




$T$:

$t_5 = 3$

$t_4 = 5$

$t_3 = 4$

$t_2 = 2$

$t_1 = 1$

$(T, \leq)$:


We must find $\pi$

$\pi = (3, 5)$

$t_{\pi(5)} = 4$

$t_{\pi(4)} = 5$

$t_{\pi(3)} = 3$

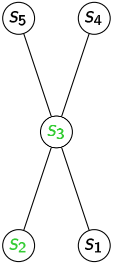
$t_{\pi(2)} = 2$

$t_{\pi(1)} = 1$

# Special case: selection

If $P := (S, \preceq)$ is a particular weak order.

$P$:




$T$:

$t_5 = 3$

$t_4 = 5$

$t_3 = 4$

$t_2 = 2$

$t_1 = 1$

$(T, \leq)$:


We must find $\pi$

$\pi = (3, 5)$

$t_{\pi(5)} = 4$

$t_{\pi(4)} = 5$

$t_{\pi(3)} = 3$

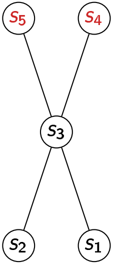
$t_{\pi(2)} = 2$

$t_{\pi(1)} = 1$

# Special case: selection

If $P := (S, \preceq)$ is a particular weak order.

$P$:




$T$:

$t_5 = 3$

$t_4 = 5$

$t_3 = 4$

$t_2 = 2$

$t_1 = 1$

$(T, \leq)$:


We must find $\pi$

$\pi = (3, 5)$

$t_{\pi(5)} = 4$

$t_{\pi(4)} = 5$

$t_{\pi(3)} = 3$

$t_{\pi(2)} = 2$

$t_{\pi(1)} = 1$

# Two more special cases

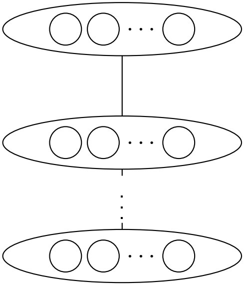
The multiple selection problem and the heap construction problem are also special cases of the partial order production problem.

# Two more special cases

The multiple selection problem and the heap construction problem are also special cases of the partial order production problem.

- If $P := (S, \preceq)$ is a weak order.

$P$:

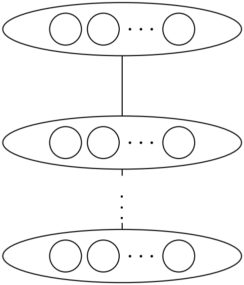


# Two more special cases

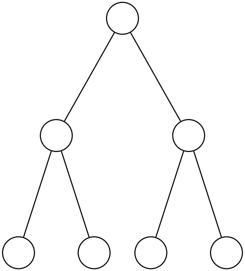
The multiple selection problem and the heap construction problem are also special cases of the partial order production problem.

- If $P := (S, \preceq)$ is a weak order.

$P$:



- If the Hasse diagram of $P := (S, \preceq)$ is complete binary tree.



# Results

Now, we can present the following theorem:

**Theorem (Cardinal, Fiorini, Joret, Jungers, Munro - 2010)**

*The Partial Order Production problem can be solved in polynomial time using at most*

$$ITLB + o(ITLB) + O(n)$$

*comparisons between elements of $T$ in the worst case.*

With $ITLB := \log n! - \log e(P)$ and $e(P)$ is the number of linear extensions of the target poset $P := (S, \preceq)$.

## Results

Now, we can present the following theorem:

**Theorem (Cardinal, Fiorini, Joret, Jungers, Munro - 2010)**

*The Partial Order Production problem can be solved in polynomial time using at most*

$$ITLB + o(ITLB) + O(n)$$

*comparisons between elements of $T$ in the worst case.*

With $ITLB := \log n! - \log e(P)$ and $e(P)$ is the number of linear extensions of the target poset $P := (S, \preceq)$.

N.B. Reading the input could take more time than necessary to solve problem.

$\Rightarrow$ Algorithms that proceed in two phases: a preprocessing phase and an ordering phase.

# Main ideas of the approach

**Idea**

The idea is to reduce P.O.P. problem to the multiple selection problem.

- We need to extend $P$ to a weak order $W$. (without increasing $ITLB$ too much)
- And then, we can use an algorithm that solves the multi-selection problem without too many comparisons. (Kaligosi *et al.*)

# Main ideas of the approach

**Idea**

The idea is to reduce P.O.P. problem to the multiple selection problem.

- We need to extend $P$ to a weak order $W$. (without increasing $ITLB$ too much)
- And then, we can use an algorithm that solves the multi-selection problem without too many comparisons. (Kaligosi *et al.*)

The analysis is made possible because $nH$ can be computed in polynomial time and provides a good estimate of $ITLB$. Here, $H$ denotes the entropy of the considered target poset and the entropy of a poset is defined as the entropy of its comparability graph.

## Main ideas of the approach

**Idea**

The idea is to reduce P.O.P. problem to the multiple selection problem.

- We need to extend $P$ to a weak order $W$. (without increasing $ITLB$ too much)
- And then, we can use an algorithm that solves the multi-selection problem without too many comparisons. (Kaligosi *et al.*)

The analysis is made possible because $nH$ can be computed in polynomial time and provides a good estimate of $ITLB$. Here, $H$ denotes the entropy of the considered target poset and the entropy of a poset is defined as the entropy of its comparability graph.

N.B. The analysis make good use of some nice results on polytopes and graphs theory.

## Analysis tools

Let $G = (E, V)$ be a perfect graph (i.e. the clic number is equal to the chromatic number for ever induced subgraph of $G$)

**Definition (Stable set polytope)**

$$\mathrm{STAB}(G) := \mathrm{conv}\{\chi^S \in \mathbb{R}^V : S \text{ stable set in G}\}$$

# Analysis tools

Let $G = (E, V)$ be a perfect graph (i.e. the clic number is equal to the chromatic number for ever induced subgraph of $G$)

**Definition (Stable set polytope)**

$$\text{STAB}(G) := \text{conv}\{\chi^S \in \mathbb{R}^V : S \text{ stable set in G}\}$$

**Definition (The entropy of a graph)**

$$H(G) := \min_{x \in \text{STAB}(G)} -\frac{1}{n} \sum_{v \in V} \log x_v$$

## The reduction

The preprocessing phase is composed of three steps:

- Apply the greedy coloring procedure to G(P) (makes use of an auxiliary network).

# The reduction

The preprocessing phase is composed of three steps:

- Apply the greedy coloring procedure to G(P) (makes use of an auxiliary network).
- Again, using the auxiliary network, extend $P$ to an interval order $I$.

# The reduction

The preprocessing phase is composed of three steps:

- Apply the greedy coloring procedure to G(P) (makes use of an auxiliary network).
- Again, using the auxiliary network, extend $P$ to an interval order $I$.
- Apply the greedy coloring procedure again, this time on $G(I)$, to obtain the weak order $W$.

This can be performed in polynomial time.

## The reduction

The preprocessing phase is composed of three steps:

- Apply the greedy coloring procedure to G(P) (makes use of an auxiliary network).
- Again, using the auxiliary network, extend $P$ to an interval order $I$.
- Apply the greedy coloring procedure again, this time on $G(I)$, to obtain the weak order $W$.

This can be performed in polynomial time.

**Remark**

It can be shown that every algorithm that first extends the target poset to a weak order and then solves the problem on the weak order can be forced to make $ITLB + \Omega(n \log \log n)$ comparisons, both in the worst case and the average case.

# Bibliography

J. Cardinal, S. Fiorini, G. Joret, R. M. Jungers, and J. I. Munro.
An effcient algorithm for partial order production.
*SIAM J. Comput., 39(7) :2927-2940, 2010.*

## Bibliography

J. Cardinal, S. Fiorini, G. Joret, R. M. Jungers, and J. I. Munro.
An effcient algorithm for partial order production.
*SIAM J. Comput., 39(7) :2927-2940, 2010.*

K. Kaligosi, K. Mehlhorn, J. I. Munro, and P. Sanders.
Towards optimal multiple selection.
*In L. Caires, G. F. Italiano, L. Monteiro, C. Palamidessi, and M. Yung, editors, ICALP, volume 3580 of Lecture Notes in Computer Science, pages 103-114.Springer, 2005.*

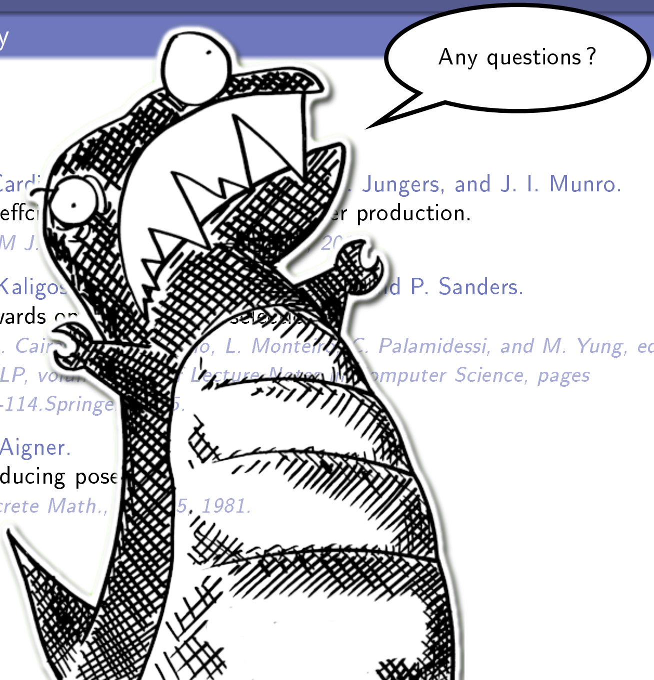
# Bibliography

J. Cardinal, S. Fiorini, G. Joret, R. M. Jungers, and J. I. Munro.
An effcient algorithm for partial order production.
*SIAM J. Comput., 39(7) :2927-2940, 2010.*

K. Kaligosi, K. Mehlhorn, J. I. Munro, and P. Sanders.
Towards optimal multiple selection.
*In L. Caires, G. F. Italiano, L. Monteiro, C. Palamidessi, and M. Yung, editors, ICALP, volume 3580 of Lecture Notes in Computer Science, pages 103-114.Springer, 2005.*

M. Aigner.
Producing posets.
*Discrete Math., 35:1-15, 1981.*

# Bibliography





J. Card ... Jungers, and J. I. Munro.
An effc ... er production.
*SIAM J. ... 20*

K. Kaligos ... d P. Sanders.
Towards o ... selectio
*In L. Cair ... o, L. Monteiro, C. Palamidessi, and M. Yung, editors, ICALP, vol ... Lecture Notes in Computer Science, pages 103-114.Springe ... 5.*

M. Aigner.
Producing pose
*Discrete Math., ... 5, 1981.*