

Exercises and solutions for the course MAT1241, Geometry II

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1 Theory

1.1 Notions of topology

Let X be a set. A **topology** on X is a choice of a set τ of parts of X such that

- $X \in \tau, \emptyset \in \tau,$
- $\bigcap_{\text{finite}} A_i \in \tau$ if $A_i \in \tau,$
- $\bigcup A_i \in \tau$ if $A_i \in \tau.$

Elements of τ are said **open** in X for the topology τ . Let us point out the difference between the second and the third point: the third point allows infinite unions. If we consider $A_i =]0, 1 + \frac{1}{i}[\subset \mathbb{R}$, we have

$$\bigcap_{i \in \mathbb{N}_0} A_i =]0, 1]$$

which is not open in \mathbb{R} .

If $E \subset X$, the **induced** topology on E by X is the one for which a subset A of E is open in E if and only if there exists a subset \mathcal{U} of X which is open in X and such that $A = E \cap \mathcal{U}$. In other words, $A \in \tau_E$ if and only if $A = E \cap \mathcal{U}$ for a certain $\mathcal{U} \in \tau_X$.

When X and Y are topological spaces, we say that $f : X \rightarrow Y$ is **continuous** if for all $A \in \tau_Y$, $f^{-1}(A) \in \tau_X$, that is, if the inverse image of any open set is open. Let us see a non trivial example.

We consider X as \mathbb{R} endowed with the usual topology and Y as \mathbb{R} with the topology $\tau_Y = \{\emptyset, \mathbb{R}\}$. We have $X = Y$ as sets, but $X \neq Y$ as topological spaces. The identity map $\text{id} : X \rightarrow Y$ is continuous because $\text{id}^{-1}(\emptyset) = \emptyset \in \tau_X$ and $\text{id}^{-1}(\mathbb{R}) = \mathbb{R} \in \tau_X$. But the identity $\tilde{\text{id}} : Y \rightarrow X$ is not continuous because $\tilde{\text{id}}^{-1}(]0, 1]) =]0, 1[$ which is not open in Y .

1.2 Differential, path and extensions

Let's take $X_0 \in T_{x_0}C$, a map $f : C \rightarrow C'$ and an extension \hat{f} of f on a neighbourhood V around x_0 . If one consider the path $\eta : \mathbb{R} \rightarrow V$, $\eta(t) = x_0 + tX_0$, we know that

$$\begin{aligned} df_{x_0}(\eta'(0)) &= \frac{d}{dt} \left[\hat{f}(\eta(t)) \right]_{t=0} \\ &= \frac{\partial \hat{f}}{\partial x^i} \frac{d\eta^i}{dt}. \end{aligned}$$

On the latter expression, we see that if $\eta(0) = \gamma(0) = x_0$ and $\gamma'(0) = \eta'(0)$, then $df_{x_0}(\gamma'(0)) = df_{x_0}(\eta'(0))$. In particular, we can chose a path γ such that $\gamma(t) \in c$ for all t . For this path, we can put $f(\gamma(t))$ in the derivative instead of $\hat{f}(\gamma(t))$.

Hence we can consider the differential of a map $f : C \rightarrow C'$ as an object intrinsically defined on C without extensions. So we set $df_{x_0} : T_{x_0}C \rightarrow T_{f(x_0)}C$,

$$df_{x_0}(\eta'(0)) = \frac{d}{dt} \left[(f \circ \eta)(t) \right]_{t=0}.$$

1.3 Vector fields

A *tangent vector field* along an embedded curve C in \mathbb{R}^2 is a rule which assigns to every point $x \in C$ a vector X_x tangent to C at x . Moreover, one requires the dependence on x “to be smooth”.

More precisely, let $T(C)$ denote the set of all vectors tangent to C :

$$T(C) = \bigcup_{x \in C} T_x(C) \quad (1)$$

where the union is a *disjoint* union. A **tangent vector field** on C is a map $X : C \rightarrow T(C)$, $x \mapsto X_x$ such that

(i) for all x , one has

$$X_x \in T_x(C), \quad (2)$$

(ii) the map $C \rightarrow \mathbb{R}^2 \times \mathbb{R}^2$,

$$x \mapsto (x, X_x) \quad (3)$$

is smooth.

The latter is a non trivial point since C is not open in \mathbb{R}^2 .

Exercise 1. Some students remarked that equation (3) was a lie because of (2): firstly $T_x C$ is vector space of dimension one (so X_x cannot belong to a \mathbb{R}^2) and secondly, the union (1) being a disjoint union, if x and y are different points of C , X_x and X_y belongs to two different spaces: $T_x C$ and $T_y C$.

Try to understand the problem and to find a solution.

1.4 Cohomology

When C is a curve in \mathbb{R}^2 , the **cohomology** of C is the quotient

$$H^1(C) = \frac{\Omega^1(C)}{dC^\infty(C)}$$

of differential forms on C by forms which are differential of some functions. When $\alpha \in \Omega^1(C)$,

$$[\omega] = \{\omega + df \mid f \in C^\infty(C)\}.$$

We are going to study two examples: $H^1(\mathbb{R})$ and $H^1(S^1)$.

1.4.1 Cohomology of \mathbb{R}

We begin by defining the form dt on \mathbb{R} by the following action of a tangent vector $\dot{\gamma}(t) = v \in \mathbb{R}$: $dt(\dot{\gamma}(t)) = v$. Each form on \mathbb{R} can be written as

$$\alpha = a dt$$

where a is a smooth function on \mathbb{R} . Why ? Because for each $x \in \mathbb{R}$, $T_x^*\mathbb{R}$ has dimension one. We are going to prove the following:

Theorem 1. *The cohomology of \mathbb{R} is zero:*

$$H^1(\mathbb{R}) = 0.$$

In other words, each form α on \mathbb{R} , reads df for a suitable function $f \in C^\infty(\mathbb{R})$. For the proof, we explicitly give a function f such that $\alpha = df$ for any $\alpha \in \Omega^1(\mathbb{R})$.

Let $\alpha \in \Omega^1(\mathbb{R})$ and let us write it as $\alpha_x = a(x) dt \in T_x^*(\mathbb{R})$. We pose

$$f(x) = \int_0^x a(u) du$$

and we compute that $df = \alpha$. Let γ be a path in \mathbb{R} ,

$$\begin{aligned} df_{\gamma(0)}\dot{\gamma}(0) &= \frac{d}{dt} \left[(f \circ \gamma)(t) \right]_{t=0} \\ &= \frac{d}{dt} \left[\int_0^{\gamma(t)} a(u) du \right]_{t=0} \\ &= a(\gamma(0))\dot{\gamma}(0). \end{aligned} \tag{4}$$

On the other hand,

$$\begin{aligned} \alpha_{\gamma(0)}\dot{\gamma}(0) &= a(x)(dt\dot{\gamma}(0)) \\ &= a(\gamma(0))\dot{\gamma}(0); \end{aligned} \tag{5}$$

this is the same !

1.4.2 Cohomology of S^1

In the whole part about S^1 , γ will denote the path

$$\gamma(t) = e^{it},$$

and σ will denote a parametrization of S^1 used in integrals.

We begin to show that $H^1(S^1) \neq 0$: there exists some 1-forms on S^1 that cannot be written under the form df . For this, we begin to prove that

Proposition 2. For all $f \in C^\infty(S^1)$,

$$\int_{S^1} df = 0.$$

Proof. If $\sigma : [0, 2\pi] \rightarrow S^1$ is a parametrization of S^1 , we have

$$\begin{aligned} \int_{S^1} df &= \int_{[0, 2\pi]} df_{\sigma(t)} \dot{\sigma}(t) dt \\ &= \int_0^{2\pi} \frac{d}{du} [(f \circ \sigma)(u)]_{u=t} dt \\ &= (f \circ \sigma)(2\pi) - (f \circ \sigma)(0) \\ &= 0. \end{aligned} \tag{6}$$

□

We define the differential 1-form α by

$$\alpha(\dot{\gamma}(t)) = 1.$$

So

$$\int_{S^1} \alpha = \int_0^{2\pi} \alpha_{\sigma(t)}(\dot{\sigma}(t)) dt = 2\pi \neq 0.$$

Then this α is not an exact form.

There exists an alternative way to define this form. We consider the function $\theta : S^1 \rightarrow \mathbb{R}$ given by

$$\theta(e^{i\eta}) = \eta. \tag{7}$$

This is the function which gives the angular coordinate of a point in S^1 . As a map from S^1 to \mathbb{R} , one can compute its differential applied on a tangent vector; for example on $\dot{\gamma}(t)$:

$$d\theta(\dot{\gamma}(t)) = \frac{d}{dt} [\theta(e^{it})]_{t=0} = \frac{d}{dt} [t]_{t=0} = 1,$$

so this is the previously defined form α . By the way, it proves that this form is a smooth differential form.

We have proved that $d\theta$ is the differential of the function θ and that it is not an exact form (because its integral is not zero). It is a problem. In fact, equation (7) doesn't *globally* defines a function on S^1 because $\theta(e^{i \cdot 0}) = 0$ but $\theta(e^{i \cdot 2\pi}) = 2\pi$. In other words, $\theta(1)$ is not well defined. Then we can only define the function θ on a local chart. For example on the half circle, we define $\theta : S^1 \rightarrow]-\frac{\pi}{2}, \frac{\pi}{2}[$ by $\theta(e^{i\sigma}) = \sigma$. This is a well defined function.

The following theorem is the main part of the proof that $H^1(S^1) = \mathbb{R}$.

Theorem 3. A form $\alpha \in \Omega^1(S^1)$ fulfil

$$\int_{S^1} \alpha = 0$$

if and only if there exists a function $f : S^1 \rightarrow \mathbb{R}$ such that $df = \alpha$.

Proposition 2 is the first half of the proof. Let's see the second one

Proof. Let $\alpha \in \Omega^1(S^1)$ such that $\int_{S^1} \alpha = 0$. It can be written under the form

$$\alpha_{e^{i\theta}} = a(e^{i\theta}) d\theta$$

and we define $f : S^1 \rightarrow \mathbb{R}$ by

$$f(e^{i\theta}) = \int_0^\theta a(e^{i\sigma}) d\sigma.$$

Since $\alpha_{e^{i\theta}}(\dot{\gamma}(\sigma)) = a(e^{i\theta}) d\theta(\dot{\gamma}(\sigma)) = a(e^{i\theta})$, in fact f is

$$f(e^{i\theta}) = \int_0^\theta \alpha. \quad (8)$$

It is important to remark that this function has not the problem of definition of the previously θ because $\int_{S^1} \alpha = 0$.

Now we want to prove that $df = \alpha$. For this we consider a path $c(t) = e^{i\theta(t)}$ in S^1 and we compute $df_{e^{i\theta(0)}}c'(0)$. We have:

$$\begin{aligned} df_{e^{i\theta(0)}}c'(0) &= \frac{d}{dt} \left[(f \circ c)(t) \right]_{t=0} \\ &= \frac{d}{dt} \left[\int_0^{\theta(t)} a(e^{i\sigma}) d\sigma \right]_{t=0} \\ &= a(e^{i\theta(0)})\theta'(0). \end{aligned} \quad (9)$$

On the other hand,

$$\begin{aligned} \alpha_{e^{i\theta(0)}}c'(0) &= a(e^{i\theta(0)}) d\theta(c'(0)) \\ &= a(e^{i\theta(0)}) \frac{d}{dt} \left[\theta(e^{i\theta(t)}) \right]_{t=0} \\ &= a(e^{i\theta(0)}) \frac{d}{dt} \left[\theta(t) \right]_{t=0} \\ &= a(e^{i\theta(0)})\theta'(0). \end{aligned} \quad (10)$$

This concludes the proof. □

It is now easy to see that

$$I : [\alpha] \mapsto \int_{S^1} \alpha$$

defines a vector space isomorphism between $H^1(S^1)$ and \mathbb{R} . We check that I is well defined: if $[\alpha] = [\beta]$, we must have $I([\alpha]) = I([\beta])$. It is correct because $[\alpha] = [\beta]$ means that there exists a function f such that $\beta = \alpha + df$. Then

$$I([\beta]) = \int_{S^1} \beta = \int_{S^1} (\alpha + df) = \int_{S^1} \alpha = I([\alpha]).$$

1.5 Integral curve

A map $c : \mathbb{R} \rightarrow \mathbb{R}^n$ is an **integral curve** of the vector field $X \in \mathfrak{X}(\mathbb{R}^n)$ if for all t (in the domain of c),

$$c'(t) = X_{c(t)}. \quad (11)$$

As a vector field, X is a map $X : \mathbb{R}^n \rightarrow \mathbb{R}^n$, so equation (11) reads

$$c'(t) = X(c(t)).$$

Now, “to be an integral curve” means “to satisfy a differential equation”. You know some existence and unicity theorems; for example this one:

Theorem 4. *Let $u : I \rightarrow \mathbb{R}$ and $f : J \rightarrow \mathbb{R}$ where J is open in \mathbb{R} and I has non empty interior. We consider the following differential equation:*

$$y'(t) = u(t)f(y(t)). \quad (12)$$

Suppose that u is continuous on I and f is continuous on J with $f(\eta) \neq 0$ for all $\eta \in J$. Let $t_0 \in I$ and $y_0 \in J$. Then there exists an interval $I' \subset I$ with non empty interior with $t_0 \in I'$ and a C^1 function $y : I' \rightarrow J$ such that

- (i) y is solution of (12) on I' and fulfils the initial condition $y(t_0) = y_0$,*
- (ii) if z is a solution of (12) in an interval $I'' \subset I$ with $t_0 \in I''$ and $z(t_0) = y_0$, then $I'' \subset I'$ and $z(t) = y(t)$ for $t \in I''$.*

When one says that an embedded curve C is an integral curve of a vector field X , one means that there exists a system of charts $\varphi_\alpha : \mathcal{U}_\alpha \rightarrow C$ on this curve which solve condition (11) as map $\varphi_\alpha : \mathcal{U}_\alpha \rightarrow \mathbb{R}^n$. In particular, X must be tangent to C on each point.

1.6 Christoffel symbols

When $f: \mathcal{U} \rightarrow \mathbb{R}^3$ is a parametrization of a surface S in \mathbb{R}^3 , there exists two interesting tangent vectors fields:

$$e_x|_{x_0, y_0} = \frac{d}{dt} \left[f(x_0 + t, y_0) \right]_{t=0} \quad \text{and} \quad e_y|_{(x_0, y_0)} = \frac{d}{dt} \left[f(x_0, y_0 + t) \right]_{t=0}. \quad (13)$$

They form a basis of $T_{f(x_0, y_0)}S$. These vectors are often called ∂_i . You can ask yourself why this partial derivative notation.

The problem of finding the first fundamental form (see exercise 13) reduce to computes products $e_i \cdot e_j$. These products form the matrix g :

$$g_{ij} := e_i \cdot e_j.$$

Now, there are no reasons for $\partial_i e_j$ to be a tangent vector. In fact, it is even difficult to *define* the object $\partial_i e_j$ because it should be something like

$$\partial_i e_j = \lim_{t \rightarrow 0} \frac{e_j(x + t) - e_j(x)}{t}.$$

But $e_j(x)$ and $e_j(x + t)$ are not vectors on the same points, therefore the difference on the right hand side is not well defined as difference in TS . This difference is however well defined in the ambient space \mathbb{R}^3 . So in order to define the derivative of a tangent vector field, we have to consider the surface as a part of \mathbb{R}^3 . In other words, we have to extend the coordinate system $(x, y) \rightarrow S$ into a coordinate system $(x, y, z) \rightarrow \mathbb{R}^3$ which reduce to original coordinates when $z = z_0$ and such that the vector e_z is normal.

As an example, polar coordinates on the (unit) sphere in \mathbb{R}^2 are given by

$$(\varphi, \theta) \rightarrow (\cos \varphi \sin \theta, \cos \varphi \cos \theta, \sin \varphi)$$

and an extension of the system is the full spherical coordinates on \mathbb{R}^3 :

$$(\varphi, \theta, r) \rightarrow (r \cos \varphi \sin \theta, r \cos \varphi \cos \theta, r \sin \varphi);$$

it is easy to see that e_r is normal. Now, our new set of e_i 's form a basis of the whole \mathbb{R}^3 , then we define the **Christoffel symbols** Γ_{ij}^k of the new coordinates system

$$\partial_i e_j = \sum_k \Gamma_{ij}^k e_k. \quad (14)$$

In this formula, k is summed over *all* coordinates, including the normal one.

There exists a funny formula to explicitly compute these Christoffel symbols. We begin by taking the derivative ∂_k of $g_{ij} = e_i \cdot e_j$, where k, i and j are

tangent coordinates. So we do not take care about the normal coordinate. Remark that in formula

$$\partial_k g_{ij} = \Gamma_{ki}^l e_l \cdot e_j + e_i \cdot \Gamma_{kj}^l e_l,$$

the summed index l take *a priori* all values including the normal coordinate, but the scalar product with e_j and e_i kill the normal component. So in the sequel we are left with only tangent coordinates. Replacing the products $e_l \cdot e_j$ and $e_i \cdot e_l$ by the corresponding elements of the matrix g ,

$$\partial_k g_{ij} = \Gamma_{ki}^l g_{lk} + \Gamma_{kj}^l g_{il}.$$

Computing the combination $\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ki}$ and taking the symmetry property $\Gamma_{ij}^k = \Gamma_{ji}^k$, $g_{ij} = g_{ji}$ into account, we find the relation

$$2\Gamma_{ki}^l g_{lj} = \partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ki}.$$

If we write $g^{ab} := (g^{-1})_{ab}$, we can rewrite it as

$$\Gamma_{ij}^k = \frac{1}{2} g^{lk} (\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}). \quad (15)$$

This is the promised formula. Now we are able to write down elements Γ_{ij}^k by simply derive and invert the matrix g .

In formula (15), indices (including the summed one) only take values in tangent coordinates. The normal one is completely forgotten.

Information Here is the definition given in the oral course. Let $r : U \subset \mathbb{R}^3 \rightarrow \Sigma \subset \mathbb{R}^3$ be a chart on an embedded surface Σ . Set $r_i := \partial_{x_i} r$; $N := \frac{1}{\|r_1 \times r_2\|} r_1 \times r_2$ ($i = 1, 2$). Then the **Christoffel symbols** Γ_{ij}^k are defined by the relations:

$$\partial_{x_i}(r_j) := \sum_k \Gamma_{ij}^k r_k + \nu_{ij} N.$$

Do you understand why the latter formula has two terms while formula (14) has only one? Is there *really* one term less in (14)?

1.7 Commutator of vector field

Let M be a surface and $X, Y \in \mathfrak{X}(M)$. We want to define $[X, Y] \in \mathfrak{X}(M)$, so for each $x \in M$ and $f \in C^\infty(M)$ we have to define $[X, Y]_x f$. We know that X_x is an element in $T_x M$ which acts on f by

$$X_x(f) = \frac{d}{dt} \left[f(X_x(t)) \right]_{t=0} = df_x X_x \in \mathbb{R},$$

so one can consider Xf as a function, $(Xf)(x) = X_x f$. We can apply the vector Y_x to this function and get a real:

$$\begin{aligned} Y_x(Xf) &= \frac{d}{dt} \left[(Xf)(Y_x(t)) \right]_{t=0} \\ &= \frac{d}{dt} \frac{d}{ds} \left[f(X_{Y_x(t)}(u)) \right]_{\substack{s=0 \\ t=0}}. \end{aligned}$$

This allows us to define $[X, Y]_x f = X_x(Yf) - Y_x(Xf)$. If, in a local coordinate system, $X = X^i \partial_i$ and $Y = Y^j \partial_j$, we have

$$\begin{aligned} (XY)f &= X^i \partial_i (Y^j \partial_j f) \\ &= X^i (\partial_i Y^j) \partial_j f + X^i Y^j \partial_{ij}^2 f. \end{aligned}$$

The same computation with $X \leftrightarrow Y$ shows that the second derivative disappears in the commutator $[X, Y]$. So $[X, Y]$ is a combination of the e_i 's and therefore is a vector fields.

The set of vector fields endowed with the bracket $[\cdot, \cdot]$ is an algebra. It is even a *Lie* algebra because the bracket

(i) is bilinear,

(ii) is skew-symmetric: $[X, Y] = -[Y, X]$,

(iii) fulfills the Jacobi identity: $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$,

for all $X, Y, Z \in \mathfrak{X}(M)$.

2 Exercises

Exercise 2. Find a smooth parametrization of the set

$$E := \{(t, |t|)\}_{t \in]-1, 1[}.$$

Is E an embedded curve in \mathbb{R}^2 ?

A **parametrization** is like a chart, but without the regularity condition.

Solution at page 15.

Exercise 3. Is the following set E an embedded curve in \mathbb{R}^2 ?

$$E := \{(t, \sin(\frac{1}{t}))\}_{t \in]0, 1[} \cup \{(0, s)\}_{s \in [-1, 1]}.$$

Solution at page 15.

Exercise 4. Consider the set

$$E = \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \theta \in \mathbb{R} \right\}.$$

- (i) Is this set an embedded curve in \mathbb{R}^4 ?
- (ii) Write a general tangent vector at $\theta = 0$. Is it surprising ?

Now, fix a $v \in \mathbb{R}^2$, and let $f_v : E \rightarrow \mathbb{R}^2$ be defined by

$$f_v(\theta) = \theta v$$

with a slight abuse of notation between θ and the element of E which is defined by θ . You have to guess the product in the right hand side.

- (iii) Give an explicit form of $(df_v)_\mathbb{1}$ where $\mathbb{1}$ is the 2×2 unit matrix.
- (iv) Can you give a geometric interpretation as “infinitesimal” rotation ? Does it help to answer point (ii) ?

Solution at page 16.

For general culture, remark that E is a group for the matrix product and that the anti-symmetric matrices form an algebra over \mathbb{R} for the product $A \cdot B := AB - BA$ where the dot denotes the algebra product and the product in the right hand side is the usual matrix product.

The elements of E depend smoothly on one parameter, turning E into what is called a **Lie group** of dimension 1. The set of antisymmetric matrices (which form the tangent space to E at the identity), endowed with the dot product, forms the **Lie algebra** associated with the Lie group E . Note that in this case, the dot product always gives zero. This is common to all Lie algebras of dimension 1.

More generally, consider a curve $x(t)$ in $O(n)$ (the group of orthogonal matrices) such that $x(0) = \mathbb{1}$. For each t , one has $x(t)x(t)^T = \mathbb{1}$. Taking the derivative of this equation with respect to t at $t = 0$, Leibnitz rule yields $\dot{x}(0)x(0)^T + x(0)\dot{x}(0)^T = 0$, or, taking $x(0) = \mathbb{1}$ into account,

$$\dot{x}(0) + \dot{x}(0)^T = 0.$$

This proves that $\dot{x}(0)$ is an antisymmetric matrix. So the Lie algebra of the Lie group of orthogonal matrices is the algebra of antisymmetric matrices. When $n \geq 3$, the algebra product $AB - BA$ is no longer identically zero.

Exercise 5. Consider the algebra $C^\infty(S^2)$ of smooth functions on the sphere S^2 and the Lie algebra $\mathfrak{X}(S^2)$ of smooth vector fields. Consider the vector field

$$Y_i = \sum_{j,k=1}^3 \epsilon_{ijk} x_j \frac{\partial}{\partial x_k}$$

with $\sum_j (x_j)^2 = 1$. Check that Y_i is actually a tangent vector and prove that these vector fields generate the whole algebra. Then, prove that this set is not free because

$$\sum_j x_j Y_j = 0.$$

This proves that the tangent space of S^2 is non trivial. Solution at page 17.

Exercise 6. Show that the set, $T(S^1)$, of all tangent vectors to $S^1 \subset \mathbb{C}$ is naturally diffeomorphic to a cylinder in \mathbb{R}^3 . Describe, in terms of the cylinder, the differential of the map $\phi : S^1 \rightarrow S^1$,

$$\phi(z) = z^N.$$

where $N \in \mathbb{N}_0$ is fixed.

Solution at page 18.

Exercise 7. Let C be an embedded curve in \mathbb{R}^2 globally traced out by a regular path $\gamma :]-1, 1[\rightarrow \mathbb{R}^2$. It means that γ is a global chart of C . Prove that the velocity field $\{\dot{\gamma}(t)\}_{t \in]-1, 1[}$ naturally defines a tangent vector field $\dot{\gamma}$ on C .

Solution at page 20.

Exercise 8. Prove that the data of a tangent vector field X on a curve C is equivalent to the data of a smooth map $\Xi : C \rightarrow \mathbb{R}^2$ with $\Xi(x) \in T_x C$, the identification being given by $\Xi(x) = X_x$. In other words, locally, a tangent vector field on a curve embedded in \mathbb{R}^2 is the restriction to C of a vector field $\hat{\Xi}$ on some open neighbourhood U of C in \mathbb{R}^2 . Draw a picture illustrating this fact.

Solution at page 21.

Exercise 9. Prove that, identifying $T(S^1)$ with the cylinder $\text{Cyl} := S^1 \times \mathbb{R} \subset \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$, the data of a tangent vector field X on S^1 is equivalent to the data of a smooth map $\xi : S^1 \rightarrow \text{Cyl}$ such that $\text{pr}_1 \circ \xi = \text{id}_{S^1}$ where $\text{pr}_1 : \mathbb{R}^3 \rightarrow \mathbb{R}^2 : (x, y, z) \mapsto (x, y)$.

Solution at page 21.

Exercise 10. Let C be a curve in \mathbb{R}^2 and $X, Y \in \mathfrak{X}(C)$ with $X_x \neq 0$ for all $x \in C$. Then there exists a smooth curve $f : C \rightarrow \mathbb{R}$ such that $Y = fX$, i.e. $Y_x = f(x)X_x$ for each $x \in C$.

Hint Show the property in the case $C = \mathbb{R}$ and $C = S^1$. In the general case, you can use a diffeomorphism to transform C into \mathbb{R} or S^1 .

Solution at page 22.

Exercise 11. Prove the following.

Lemma 5. *Let X be a smooth non vanishing vector field on an open set \mathcal{U} in \mathbb{R}^2 , $f : \mathcal{U} \rightarrow \mathbb{R}$ a smooth non vanishing function and Y the smooth vector field on \mathcal{U} defined by $Y_x = f(x)X_x$. If $c : I \rightarrow \mathbb{R}^2$ is an integral curve of X , then there exists a smooth map $\alpha : I \rightarrow \mathbb{R}$ such that $c \circ \alpha$ is an integral curve of Y .*

In other words, if one changes the norm of a vector field, then one just has to change the parametrization of integral curves.

Solution at page 24.

Exercise 12. In the setting of exercise 8, prove that the curve C is an **integral curve** of $\hat{\Xi}$ i.e. for all initial data $x_0 \in C$ the solution $x = x(t)$, $t \in]-\epsilon, \epsilon[$ to the Cauchy problem

$$\dot{x} = \hat{\Xi}; \quad x(0) = x_0$$

is a local chart on C around x_0 . We suppose that X is non vanishing.

Solution at page 24.

Hints Draw a circle and any smooth tangent vector field. Extend it in any smooth manner. Then pick a point on the circle. Is it possible to find an integral curve of your vector field which is not the circle? Try and convince yourself that the integral curves of a vector field X are defined by the field of *directions* and are independent of the *norm* of X_x .

Exercise 13. Prove that on an embedded surface $\Sigma \subset \mathbb{R}^3$, the following formula

$$\mathcal{I}_x(v, w) := v \cdot w$$

for each $x \in \Sigma$ and $v, w \in T_x(\Sigma) \subset \mathbb{R}^3$ defines a Riemannian metric, \mathcal{I} , on Σ called the **first fundamental form** of Σ .

Solution at page 25.

Exercise 14. Give the expression of the first fundamental form of the sphere S^2 :

- (i) within spherical coordinates (θ, ϕ) ;
- (ii) within stereographical coordinates $z \in \mathbb{C}$.

Solution at page 25.

Exercise 15. Compute the first fundamental form of the torus $T^2 \subset \mathbb{R}^3$.
Solution at page 25.

Exercise 16. Same question for a surface of revolution generated by the curve $[y = f(x)] \cap [z = 0]$. Solution at page 26.

Exercise 17. Compute the Christoffel symbols in the following cases: the plane $XoY \subset \mathbb{R}^3$, the sphere parametrized by spherical coordinates (θ, ϕ) , a surface of revolution.

Solution at page 28.

Exercise 18. Since basis vectors given in equation (13) are just derivatives with respect to coordinates, it is obvious that they commute: $[e_i, e_j] = 0$. The fact that basis vectors commute seems to prove that any vector field should commute. Find the mistake. Can you find two vector fields on the sphere which doesn't commute ?

Solution at page 28.

3 Solutions

Solution of exercise 2

The parametrization given in the question is not smooth because the derivative is discontinuous at $t = 0$. Remark that tangent vectors in this parametrization must be parallel to $(1, -1)$ in the left part and to $(1, 1)$ in the right part. The only way for a function (namely the vertical part of the derivative of the parametrization) to change sign without discontinuity is to vanish. One thus needs a parametrization whose derivative vanishes at $(0, 0)$. This already answers the second question: this is not an embedded curve.

Consider the parametrization $\varphi(t) = (t^2, x(t))$ where

$$x(t) = \begin{cases} -t^2 & \text{when } t \in]-1, 0] \\ t^2 & \text{when } t \in]0, 1[. \end{cases}$$

This parametrization doesn't work because the second derivative has a discontinuity at $t = 0$. Try to convince yourself that a correct parametrization is given by

$$\varphi(t) = \begin{cases} (1, -1)f(t) & \text{if } t \in]-1, 0] \\ (1, 1)f(t) & \text{if } t \in]0, 1[\end{cases}$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function which gives 1 at 1 and -1 and vanishes at zero with all its derivatives. Such a function can be built from Cauchy's regularization function $\rho(x) = e^{-1/x^2}$.

The annihilation of *all* derivatives of the parametrization at $(0, 0)$ is required by the fact that on the left, a derivative of any order must be a multiple of $(1, -1)$ while on the right it must be a multiple of $(1, 1)$.

Solution of exercise 3

Let us denote $E_1 = \{(t, \sin(\frac{1}{t}))\}_{t \in]0, 1[}$ and $E_2 = \{(0, s)\}_{s \in [-1, 1]}$. If one wants E to be an embedded curve in \mathbb{R}^2 , one has to find maps $\varphi_\alpha : \mathcal{U}_\alpha \rightarrow E$ such that the famous three conditions hold.

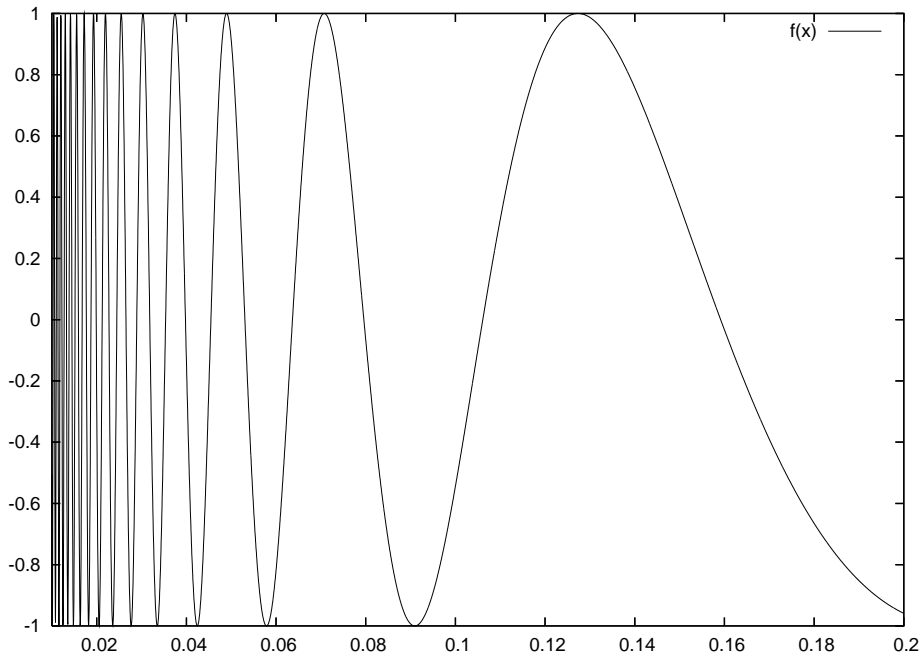


Figure 1: The function $t \rightarrow \sin(\frac{1}{t})$ powered by Gnuplot

One of these maps must contain the point $a = (0, -1)$; let's say $\varphi(1) = a$. Consider the open set $B(a, r)$ of radius $r < \frac{1}{2}$ around a . If φ is continuous, there exists a δ such that $\epsilon \leq \delta$ implies $\varphi(1 + \epsilon) \in B(a, r)$.

It is possible to find an open neighbourhood B' of a which does not contain the oscillation on which b lies. Let $b' = \varphi(1 + \epsilon') \in B'$. Since φ is continuous, the path $c: [\epsilon', \epsilon] \rightarrow C$ given by $c(t) = \varphi(1 + t)$ is continuous (as path in \mathbb{R}^2). A point of the image of c has the form $\varphi(1 + s)$ with $s \leq \epsilon \leq \delta$. So $c([\epsilon', \epsilon]) \subset B(a, r)$.

But the fact that c reaches b' from b which is not on the same oscillation forces at least one point of c to be on the top of an oscillation and then to get out from $B(a, r)$. This contradicts the continuity.

Solution of exercise 4

The natural mistake is to say “ θ takes values in $[0, 2\pi[$ which is not an open interval. Then I cannot parametrize E ”. One can use more than one local chart ! For instance it is possible to take

$$\begin{aligned} \mathcal{U}_1 &=]0, 2\pi[, \\ \mathcal{U}_2 &=]\frac{3\pi}{2}, \frac{5\pi}{2}[\end{aligned}$$

and in both cases

$$\varphi_i(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

In order to write a tangent vector at $\theta = 0$, we consider a general path in E given by a path $\theta : \mathbb{R} \rightarrow \mathbb{R}$ such that $\theta(0) = 0$. We have

$$\begin{aligned} \frac{d}{dt} \left[\begin{pmatrix} \cos \theta(t) & \sin \theta(t) \\ -\sin \theta(t) & \cos \theta(t) \end{pmatrix} \right]_{t=0} &= \begin{pmatrix} -\theta'(0) \sin \theta(0) & \theta'(0) \cos \theta(0) \\ -\theta'(0) \cos \theta(0) & -\theta'(0) \sin \theta(0) \end{pmatrix} \\ &= \theta'(0) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned} \quad (16)$$

which is a general anti-symmetric matrix. Formally, you have to interpret it as a vector in \mathbb{R}^4 , but matrix realization is fruitful to interpret the sequel.

Now the map $(df_v)_1$ has to be applied to a tangent vector at point $\mathbb{1}$, so one computes (we set $v = (v_x, v_y)$)

$$\begin{aligned} (df_v)_1 \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} &= \frac{d}{dt} \left[f_v \begin{pmatrix} \cos at & \sin at \\ -\sin at & \cos at \end{pmatrix} \right]_{t=0} \\ &= \frac{d}{dt} \left[\begin{pmatrix} v_x \cos at + v_y \sin at \\ -v_x \sin at + v_y \cos at \end{pmatrix} \right]_{t=0} \\ &= a \begin{pmatrix} v_y \\ -v_x \end{pmatrix}. \end{aligned} \quad (17)$$

If you apply a “very little” rotation on the vector $\begin{pmatrix} v_x \\ v_y \end{pmatrix}$, the displacement is given by the vector $\begin{pmatrix} v_y \\ -v_x \end{pmatrix}$.

Solution of exercise 5

Let us pick a point $x = (x_1, x_2, x_3) \in S^2$ and check that the vector $Y_i(x)$ is tangent to S^2 at the point x . For that, we just have to prove that the product of $Y_i(x)$ and x is zero (when you are tangent to a sphere, you are perpendicular to the radius):

$$Y_i(x) \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \sum_{j,k=1}^3 \epsilon_{ijk} x_j x_k = 0$$

because contraction of symmetric indices (jk in $x_j x_k$) and skew-symmetric indices (jk in ϵ_{ijk}) is zero.

Solution of exercise 6

A point $z \in S^1$ can be written under the form $z = e^{i\theta_0}$. A tangent vector of S^1 at the point z can be expressed by means of the derivative of a path $X : \mathbb{R} \rightarrow S^1$ such that $X(0) = z$. Let us consider a general path $X(t) = e^{i\theta(t)}$ with $\theta_0 = \theta_0$. Its tangent vector is

$$X = \frac{d}{dt} \left[e^{i\theta(t)} \right]_{t=0} = i\theta'(0)e^{i\theta_0} = \theta'(0)e^{i\left(\frac{\pi}{2} + \theta_0\right)}. \quad (18)$$

Let us point out the fact that the velocity $\theta'(0)$ of the path describes the norm of the tangent vector while $e^{i\theta_0}$ is the point of S^1 on which X is fixed. The differential of ϕ at z on the vector X reads

$$\begin{aligned} d\phi_z X &= \frac{d}{dt} \left[\phi(X(t)) \right]_{t=0} \\ &= \frac{d}{dt} \left[e^{iN\theta(t)} \right]_{t=0} \\ &= N\theta'(0)e^{i(N-1)\theta_0} e^{i\left(\frac{\pi}{2} + \theta_0\right)} \\ &= Nz^{N-1}X. \end{aligned} \quad (19)$$

Now let us see TS^1 as subset of \mathbb{R}^4 . Any $X \in TS^1$ can be written as (z, v) where $z \in S^1 \subset \mathbb{C} \simeq \mathbb{R}^2$ is the point at which X is “fixed” and $v \in \mathbb{C} \simeq \mathbb{R}^2$ is the vector itself. Relation (18) contains strong constraints on elements $(z, v) \in \mathbb{R}^4$: if $z = e^{i\theta_0} = (\cos \theta, \sin \theta)$, the element v must be of the form $he^{i\left(\frac{\pi}{2} + \theta_0\right)} = (-h \sin \theta, h \cos \theta)$. Now TS^1 is seen as the subset of \mathbb{R}^4 of vectors of the form

$$(z, X) = (\cos \theta, \sin \theta, -h \sin \theta, h \cos \theta).$$

At this point it is important to emphasize two points.

- The map h can be negative as well as positive.
- An element in \mathbb{C} does not uniquely determines an element in TS^1 although tangent spaces of $e^{i\theta}$ and $e^{i(\pi+\theta)}$ look like the same (draw a picture if you are unsure). When one deals with a tangent vector, one has to keep trace of the point on which the vector is fixed. If one does not, one can believe that $T_{z_1}S^1 = T_{z_2}S^1$ and then that finally, $TS^1 = \mathbb{C}$, see figure 2.

The way to see a cylinder (of radius 1) as a subset of \mathbb{R}^3 is to take the coordinates $(\cos \theta, \sin \theta, h)$. A diffeomorphism $f : TS^1 \rightarrow \text{Cyl}$ is given by

$$f(\cos \theta, \sin \theta, -h \sin \theta, h \cos \theta) = (\cos \theta, \sin \theta, h). \quad (20)$$

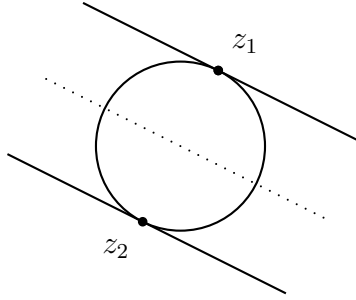


Figure 2: A possible mistake is to confuse $T_{z_1}S^1$ and $T_{z_2}S^1$.

We have to check that it is diffeomorphic: it is smooth, bijective and the inverse is smooth too. This needs the “new” definition of smooth map. Indeed the map f is defined on a rather non trivial subset of \mathbb{R}^4 : the set of (x, y, u, v) such that

$$x^2 + y^2 = 1 \quad \text{and} \quad \begin{cases} v = -ux/y & \text{if } y \neq 0 \\ v \in \mathbb{R} & \text{if } y = 0 \end{cases}$$

which is not open. We have to consider pointwise a neighbourhood and a prolongation. Let's take the part $A_0 \equiv x \neq 0, y \neq 0$ of the domain of f . On this part of the domain, f reads $(x, y, u, v) \mapsto (x, y, \frac{v}{x})$. We can find an open set $A \subset \mathbb{R}^4$ which contains A_0 and such that x and y don't vanish on A . The prolongation of f from A_0 to A by the same formula $(x, y, u, v) \mapsto (x, y, \frac{v}{x})$ is a smooth map.

When $x = 0$, the map f reads $(0, 1, 0, h) \mapsto (1, 0, h)$ which can be prolonged to the smooth map $(x, y, z, h) \mapsto (y, x, h)$. The same idea is true on $y = 0$ where f reads $(1, 0, 0, h) \mapsto (1, 0, h)$.

The inverse map f^{-1} is $(x, y, h) \mapsto (x, y, -hy, hx)$ with the constraint $x^2 + y^2 = 1$ on the domain.

The following diagram is the *definition* of the dotted line; this is what one means when one says “ $d\phi$ seen on the cylinder”:

$$\begin{array}{ccc} p \in \text{Cyl} & \xrightarrow{f^{-1}} & f^{-1}(p) \in TS_z^1 \\ \downarrow d\phi & & \downarrow d\phi_z \\ (f \circ d\phi \circ f^{-1})(p) \in \text{Cyl} & \xleftarrow{f} & d\phi_z(f^{-1}(p)) \in T_{\phi(z)}S^1. \end{array}$$

In this, we suppose that $p \in \text{Cyl}$ is the image by f of a tangent vector on S^1 at the point z .

In order to express the $d\phi_z$ of equation (19) in terms of the cylinder, we pick $X = (-\sin \theta, \cos \theta, r) \in \text{Cyl}$, i.e. $X = he^{i(\frac{\pi}{2}+\theta)} \in T_{e^{i\theta}}S^1$ and we apply $d\phi_z$ on it:

$$\begin{aligned} d\phi_z \left(he^{i(\frac{\pi}{2}+\theta)} \right) &= Ne^{i(N-1)\theta} he^{i(\frac{\pi}{2}+\theta)} \\ &= Nhe^{i(\frac{\pi}{2}+N\theta)} \\ &= (-\sin N\theta, \cos N\theta, Nh). \end{aligned}$$

Solution of exercise 7

For each t_0 , it is clear that $\dot{\gamma}(t_0) \in T_{t_0}C$. The problem is to prove that the function $f : C \rightarrow \mathbb{R}^2 \times \mathbb{R}^2$, $f(x) = (x, X_x)$ is smooth. Let x be a point of C and \mathcal{U} an open neighbourhood of x . We have to find a smooth extension $\tilde{f} : \mathcal{U} \rightarrow \mathbb{R}^2 \times \mathbb{R}^2$ of f . In particular, we have to find an extension \tilde{X} of X in a neighbourhood \hat{C} of C in \mathbb{R}^2 .

First method Since γ is a chart, it can be extended to a smooth bijection $\tilde{\gamma} :]-1, 1[\times]-1, 1[\rightarrow \mathcal{U}$ such that $\tilde{\gamma}(0, t) = \gamma(t)$. For each $y = \tilde{\gamma}(u_0, t_0) \in \mathcal{U}$, we define

$$\tilde{X}_y = \frac{d}{dt} \left[\tilde{\gamma}(u, t) \right]_{t=t_0}.$$

Now we define $\tilde{f}(y) = (y, \tilde{X}_y)$. The vector field \tilde{X} being a directional derivative of a smooth function, it is itself a smooth function.

Here is a partial proof of the existence of the extension $\hat{\gamma}$.

Lemma 6. *Let φ be a chart of an embedded curve C in \mathbb{R}^2 ; it is seen as a map $\varphi : \mathcal{U} \rightarrow \hat{C}$ where \hat{C} is an open neighbourhood of C in \mathbb{R}^2 .*

Then there exists an open set $W \subset \mathbb{R}^2$ and a smooth map $\varphi_2 : W \rightarrow \hat{C}$ such that $\tilde{\varphi} = \varphi_2^{-1} \circ \varphi : \mathcal{U} \rightarrow W$ has the form

$$\tilde{\varphi}(x) = (x, 0).$$

We will not prove it here.

Now we can extend this $\tilde{\varphi}$ to the map $\hat{\varphi} : \mathcal{U} \times I \rightarrow W$ by $\hat{\varphi}(x, y) = (x, y)$ for a suitable neighbourhood I of 0 in \mathbb{R} . Then we look at

$$\hat{\varphi} = \varphi_2 \circ \hat{\varphi} : \mathcal{U} \times I \rightarrow \hat{C}$$

which is smooth.

How to choose I ? For each $x \in \mathcal{U}$, the point $(x, 0)$ belongs to the open set $W \subset \mathbb{R}^2$. Therefore there exists a non empty neighbourhood I_x of 0 such

that $(x, I_x) \subset W$. Taking if necessary a smallest chart \mathcal{U}' instead of \mathcal{U} , one can take the non empty set $I = \bigcap_{x \in \mathcal{U}} I_x$.

Solution of exercise 8

Let us consider a smooth map $\Xi : C \rightarrow \mathbb{R}^2$ with $\Xi(x) \in T_x(C)$. It defines the tangent vector field $X_x = \Xi(x)$. It is smooth because Ξ is, and thus so is $x \mapsto (x, X_x)$. The inverse sense is the same.

Solution of exercise 9

A general form of a function $\xi : S^1 \rightarrow \text{Cyl}$ is

$$\xi(x, y) = (x, y, h(x, y))$$

only defined on $x^2 + y^2 = 1$. Here, h is a smooth function. We naturally associate to ξ the field

$$X_{(x,y)} = h(x, y)e^{i\left(\frac{\pi}{2} + \theta\right)}$$

where θ is the angular polar coordinate of (x, y) . We have to prove that this field is smooth. For this, we have to prove that

$$(x, y) \rightarrow (x, y, h(x, y)e^{i\left(\frac{\pi}{2} + \theta\right)}) \quad (21)$$

is smooth at each point of $x^2 + y^2 = 1$. Since h is smooth, by definition it can be extended. So we can extend equation (21) keeping the same form around any $(x_0, y_0) \in S^1$. The result is smooth in the sense of functions from an open set of \mathbb{R}^2 to an open set in \mathbb{R}^3 .

Let us now prove that a vector field defines a function. The general form of a vector field on S^1 is

$$X_{(x,y)} = h(x, y)e^{i\left(\frac{\pi}{2} + \theta\right)} \quad (22)$$

where h is smooth on S^1 and can thus be extended into a smooth function on a neighbourhood of each point of S^1 . The map $\xi : S^1 \rightarrow \text{Cyl}$ we were looking for is thus

$$\xi(x, y) = (x, y, h(x, y)).$$

A second way is possible The reader should remark that the construction given up to here is essentially the construction of exercise 6. We can directly exploit the diffeomorphism $f : TS^1 \rightarrow \text{Cyl}$. So in a first time, we pick a $\xi : S^1 \rightarrow \text{Cyl}$ and we want to build a vector field X^ξ , i.e. a map $X^\xi : S^1 \rightarrow TS^1$. The natural candidate is

$$X^\xi = f^{-1} \circ \xi.$$

On the one hand, smoothness comes from smoothness of f^{-1} and ξ and on the other hand, $X^\xi(x) \in T_x S^1$ because of projection property of ξ .

In a second time, we pick a vector field $X : S^1 \rightarrow TS^1$ and we want to build a map $\xi^X : S^1 \rightarrow \text{Cyl}$. The candidate is

$$\xi^X = f \circ X.$$

You have to check that $\text{pr}_1 \circ \xi^X = \text{id}$; it comes from construction of f .

Now in order to say that the data of the function is “the same” that the data of the vector field, we have to prove that the two constructions are inverse each other. In other words, you are now able to do two things: if I give you a vector field, you can give me a function and if I give you a function, you can give me a vector field. Let me give you the function ξ . So you give me a vector field X^ξ . Now I give you X^ξ ; will you give back the original function ξ ? If not, the whole construction has no interest.

$$\xi \rightarrow X^\xi \rightarrow \xi^{X^\xi} \stackrel{?}{=} \xi.$$

From definitions, $\xi^{X^\xi} = f \circ X^\xi = f \circ f^{-1} \circ \xi = \xi$. This is a good point. The other is

$$X \rightarrow \xi^X \rightarrow X^{\xi^X} \stackrel{?}{=} X;$$

from definitions, $X^{\xi^X} = f^{-1} \circ \xi^X = f^{-1} \circ f \circ X = X$.

Solution of exercise 10

When $C = \mathbb{R}$, a vector field is just a smooth function $\mathbb{R} \rightarrow \mathbb{R}$ (be sure that you deeply understand this assertion!). If g_X is the function of the vector field X and f_Y the one of Y , the answer is given by

$$f(x) = \frac{g_Y(x)}{g_X(x)} \tag{23}$$

which is smooth as quotient of smooth functions.

We turn our attention to the case $C = S^1$. The formula, for each $x \in C$

$$Y_x = f(x)X_x$$

completely defines f because X doesn't vanishes. The problem is to prove that it is smooth (extensions and all that 😞). The natural way to begin is to say that X and Y admit smooth extensions \hat{X} and \hat{Y} in terms of which, we should be able to build an extension \hat{f} . The problem is that you have no guarantee that \hat{X} and \hat{Y} are parallel outside C .

One can build normal vectors to the curve and translate X and Y along these lines. Then one gets smooth parallel extensions. Such a construction

was given in the course during the proof of the lemma just below the inverse function theorem. From there, one defines the extension of f by

$$\hat{Y}_z = \hat{f}(z)\hat{X}_z$$

for all z in a neighbourhood of C . But how to prove that it is smooth? Maybe it is possible. I don't know, but this formula seems to define a smooth function since f is a quantity which relates two smooth quantities each other.

A possible way to solve the exercise is to use results of exercise 9. Let $\xi_X : S^1 \rightarrow \text{Cyl}$ be the map associated to X , and ξ_Y the one associated with Y . They can be written under the form

$$\xi_X(x) = (\cdot, \cdot, h_X(x)) \tag{24a}$$

$$\xi_Y(x) = (\cdot, \cdot, h_Y(x)) \tag{24b}$$

where the dots represent some non essential functions. The functions h_X and h_Y are smooth and we define

$$\hat{f}(z) = \frac{\hat{h}_Y(z)}{\hat{h}_X(z)}$$

which is smooth and a correct extension of f .

Now a general curve C is diffeomorphic of \mathbb{R} or S^1 . We denote by $\alpha : C \rightarrow E$ the diffeomorphism where E denotes \mathbb{R} or S^1 . The map α allows us to transform a complicated problem on C into a solved problem on E . We consider $\tilde{X} = d\alpha X$ and $\tilde{Y} = d\alpha Y$, two vector fields on E . Then there exists $\tilde{f} : E \rightarrow \mathbb{R}$ such that

$$\tilde{Y}_x = \tilde{f}(x)\tilde{X}_x \tag{25}$$

for each $x \in E$.

Let us summarize the functions that we have at hand:

$$\alpha : C \rightarrow E$$

$$\tilde{f} : E \rightarrow \mathbb{R}$$

and we are searching for

$$f : C \rightarrow \mathbb{R}.$$

Obviously the candidate is $f = \tilde{f} \circ \alpha$ which is smooth because it is a composition of smooth functions. In order to check that it is the right function, we have to prove that, for all $x \in C$, $Y_x = (f \circ \alpha)(x)X_x$.

The defining property of \tilde{f} is

$$d\alpha_x Y_x = \tilde{f}(\alpha(x)) d\alpha_x X_x.$$

Since α is diffeomorphic, $d\alpha_x$ is a linear injective map for each x ; so we can “simplify” both sides by $d\alpha_x$.

Solution of exercise 11

By definition of c as an integral curve of X , we have

$$\dot{c}(t) = X_{c(t)}$$

for all t . The requirement for $c \circ \alpha$ to be an integral curve of Y reads

$$\begin{aligned} (c \circ \alpha)'(t) &= c'(\alpha(t))\alpha'(t) \\ &\stackrel{!}{=} Y_{(c \circ \alpha)(t)} \\ &= f((c \circ \alpha)(t))X_{(c \circ \alpha)(t)}. \end{aligned} \tag{26}$$

Since we supposed that X is non vanishing, $c'(t) \neq 0$ everywhere. So we find the following differential equation for α :

$$\alpha'(t) = (f \circ c)(\alpha(t)). \tag{27}$$

The existence of a solution for this equation is given by the following powerful theorem 4.

Remark that this theorem only deals with continuity (not smoothness). It should be an exercise to find an even more powerful theorem which ensures the smoothness of α .

Solution of exercise 12

Let us begin by *almost* true proof. Why “almost” ? Because we will not care about *locality* problems. We will speak about C , \mathbb{R}^2, \dots while we should speak about “a neighborhood of $x_0 \in C$ ” ... You should do it as an exercise.

Let $\varphi : \mathcal{U} \rightarrow C$ be a parametrization of C with tangent vector field V . We want to find a new parametrization $\phi : \mathcal{U}' \rightarrow C$ with tangent vector field $X_x = f(x)V_x$ where the smooth function f is given. In other words, we want a $a : \mathcal{U}' \rightarrow \mathcal{U}$ and $\phi = \varphi \circ a$ with

$$\phi'(t) = (\varphi \circ a)'(t) = \varphi'(a(t))a'(t).$$

The function a is the solution of the differential equation

$$a'(t) = f(a(t)).$$

So another way to state this result is to say that there always exists a reparametrization of a curve for which the tangent vector field is given.

Solution of exercise 13

The fact for \mathcal{I} to be bilinear, symmetric, positive definite and nondegenerate comes from the corresponding properties of the inner product on \mathbb{R}^3 . The eventual problem is smoothness. As usual, we have to find extensions. Let $X, Y \in \mathfrak{X}(\Sigma)$; the problem is to find a (local) smooth extension of the function $\mathcal{I}(X, Y) : \Sigma \rightarrow \mathbb{R}, x \mapsto \mathcal{I}_x(X_x, Y_x) = X_x \cdot Y_x$. If \hat{X} and \hat{Y} are extensions of X and Y , what do you think about

$$\hat{\mathcal{I}}(X, Y)(y) = \hat{X}_y \cdot \hat{Y}_y \quad ?$$

Solution of exercise 14

The chart $f : \mathbb{R}^2 \rightarrow S^2$ of S^2 that we chose is

$$f(\theta, \varphi) = (\cos \theta \sin \varphi, \cos \theta \cos \varphi, \sin \theta).$$

In these coordinates, a basis of $T_{(\theta_0, \varphi_0)}S^2$ is given by

$$\frac{d}{dt} \left[f(\theta_0 + t, \varphi_0) \right]_{t=0} \quad \text{and} \quad \frac{d}{dt} \left[f(\theta_0, \varphi_0 + t) \right]_{t=0}.$$

Computations give

$$\partial_\theta = - \begin{pmatrix} \sin \theta_0 \sin \varphi_0 \\ \sin \theta_0 \cos \varphi_0 \\ -\cos \theta_0 \end{pmatrix}, \quad (28a)$$

$$\partial_\varphi = \begin{pmatrix} \cos \theta_0 \cos \varphi_0 \\ -\cos \theta_0 \sin \varphi_0 \\ 0 \end{pmatrix}. \quad (28b)$$

One can check that $\partial_\theta \cdot f(\theta_0, \varphi_0) = \partial_\varphi \cdot f(\theta_0, \varphi_0) = 0$: tangent vectors are orthogonal to radius. It is easy to see that $\partial_\varphi \cdot \partial_\theta = 0$, $\partial_\theta \cdot \partial_\theta = 1$ and $\partial_\varphi \cdot \partial_\varphi = \cos^2 \theta$.

Solution of exercise 15

Let us parametrize the torus by two angles θ (radius R) and φ (radius r). We suppose that θ describes a circle in the plane xz (or a parallel plane when φ is non zero) while the plane of the rotation φ depends of θ .

A simple rotation (an isometry of \mathbb{R}^3) allows us to only consider points with $\theta = 0$. So we expect that the result will not depend of θ . Let us

explicitly give the parametrization $f : \mathbb{R}^2 \rightarrow T^2$. When φ_0 is fixed, the curve $f(\theta, \varphi_0)$ is a circle of radius $R + r \cos \varphi_0$ in a plane parallel to xz with center at $(0, r \sin \varphi_0, 0)$. Thus we have

$$f(\theta, \varphi) = \left((R + r \cos \varphi) \sin \theta, r \sin \varphi, (R + r \cos \varphi) \cos \theta \right). \quad (29)$$

We find

$$\partial_\varphi|_{(\theta_0, \varphi_0)} \frac{d}{dt} \left[f(\theta_0, \varphi + t) \right]_{t=0} = -r \left(\sin_0 \varphi \sin \theta_0, -\cos \varphi_0, \sin \varphi_0 \cos \theta_0 \right) \quad (30a)$$

$$\partial_\theta|_{(\theta_0, \varphi_0)} = \frac{d}{dt} \left[f(\theta_0 + t, \varphi_0) \right]_{t=0} = (R + r \cos \varphi_0) \left(\cos \theta_0, 0, -\sin \theta_0 \right). \quad (30b)$$

Given under a matrix form, the result is

$$g = \begin{pmatrix} r^2 & 0 \\ 0 & (R + r \cos \varphi)^2 \end{pmatrix}.$$

Can you geometrically understand the dependance in r , R and φ ?

Note that the norm of an angular coordinate is the radius of the corresponding circle.

Solution of exercise 16

We parametrize the surface by x and θ :

$$f(x, \theta) = (x, f(x) \cos \theta, f(x) \sin \theta).$$

In this parametrization, the basis vectors are given by

$$e_x = (1, f'(x) \cos \theta, f'(x) \sin \theta) \quad (31a)$$

$$e_\theta = (0, -f(x) \sin \theta, f(x) \cos(\theta)), \quad (31b)$$

and the matrix g is given by

$$g = \begin{pmatrix} 1 + f'(x)^2 & 0 \\ 0 & f(x)^2 \end{pmatrix}$$

Christoffel symbols are obtained by formula

$$\Gamma_{ij}^k = \frac{1}{2} g^{lk} (\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij})$$

where the g^{lk} are elements of the *inverse* matrix of g . The non zero symbols are

$$\Gamma_{\theta\theta}^x = -\frac{ff'}{1+f'^2}, \quad \Gamma_{\theta x}^\theta = \Gamma_{x\theta}^\theta = \frac{f'}{f}, \quad \Gamma_{xx}^x = \frac{f'f''}{1+f'^2}. \quad (32)$$

Covariant derivatives of basis vectors are

$$\nabla_x e_x = \Gamma_{xx}^x e_x \qquad \nabla_\theta e_x = \Gamma_{\theta x}^\theta e_\theta \quad (33)$$

$$\nabla_x e_\theta = \Gamma_{x\theta}^\theta e_\theta \qquad \nabla_\theta e_\theta = \Gamma_{\theta\theta}^x e_x. \quad (34)$$

As far as Riemann tensor and related curvature issues are concerned, we do not need to compute $\nabla_x \nabla_x e_i$ and $\nabla_\theta \nabla_\theta e_i$. Computations are as follows:

$$\begin{aligned} \nabla_x \nabla_\theta e_x &= \partial_x (\Gamma_{\theta x}^\theta) e_\theta + \Gamma_{x\theta}^\theta \Gamma_{x\theta}^\theta e_\theta \\ &= \frac{f''}{f} e_\theta. \end{aligned}$$

Other results —up to personal faults— are

$$\begin{aligned} \nabla_x \nabla_\theta e_\theta &= \left(\frac{3ff'^2 f''}{(1+f'^2)^2} = \frac{f'^2 + ff''}{1+f'^2} \right) e_x \\ \nabla_\theta \nabla_x e_x &= \left(\frac{f'f''}{1+f'^2} \right) \frac{f'}{f} e_\theta \\ \nabla_\theta \nabla_x e_\theta &= -\frac{f'}{f} \frac{ff'}{1+f'^2} e_x. \end{aligned} \quad (35)$$

Isometry criterion All this part is “without proof”.

We consider the following tensor:

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$$

and the **Ricci curvature** pointwise defined by

$$r(Y, Z) = \text{Tr} [X \mapsto R(X, Y)Z].$$

It is a trace in the sense of the trace of a matrix of a linear operator in a vector space: $R(\cdot, Y)Z: T_x \Sigma^f \rightarrow T_x \Sigma^f$. For a general $A: V \rightarrow V$, it is defined by

$$\text{Tr} A = \sum_i \langle e_i, Ae_i \rangle$$

where $\{e_i\}$ is a basis of V . We define the **scalar curvature** by

$$\rho = \sum_{a=1}^2 r(e_a, e_a).$$

This defines a real valued function $\rho^f: \Sigma^f \rightarrow \mathbb{R}$ for the surface Σ^f and a corresponding one $\rho^g: \Sigma^g \rightarrow \mathbb{R}$ for Σ^g . From symmetry of the problem, these functions do not depend on the angular part of the parametrization of Σ^f and Σ^g . So one can only consider the restrictions

$$\rho^f:]a, b[\rightarrow \mathbb{R}, \quad (36a)$$

$$\rho^g:]c, d[\rightarrow \mathbb{R}. \quad (36b)$$

The criterion for Σ^f and Σ^g to be isometric is the existence of a diffeomorphism $\varphi:]a, b[\rightarrow]c, d[$ such that

$$\rho^f = \rho^g \circ \varphi.$$

Explicit proof of existence of such a φ can be very hard.

Solution of exercise 17

The polar coordinate on the sphere is given by

$$f(\theta, \varphi) = r \begin{pmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ \sin \theta \end{pmatrix}$$

where r is the (constant) radius. Simple derivations shows that

$$e_\theta = r \begin{pmatrix} -\sin \theta \cos \varphi \\ -\sin \theta \sin \varphi \\ \cos \theta \end{pmatrix}, \quad e_\varphi = r \begin{pmatrix} -\cos \theta \sin \varphi \\ \cos \theta \cos \varphi \\ 0 \end{pmatrix}.$$

The metric matrix is given by

$$g = r^2 \begin{pmatrix} 1 & \\ & \cos^2 \theta \end{pmatrix}.$$

The non zero Christoffel symbols are

$$\begin{aligned} \Gamma_{\varphi\varphi}^\theta &= \cos \theta \sin \theta, \\ \Gamma_{\theta\varphi}^\varphi &= -\tan \theta. \end{aligned} \quad (37)$$

Solution of exercise 18

When coefficient of a vector *field* are non constant functions, the computation of XYf shows that X act not only on f but also on the coefficients of

X . For example $X = e_\theta$ and $Y = \sin \theta e_\varphi$. At (θ_0, φ_0) , the path which define these vectors are

$$Y_{(\theta_0, \varphi_0)}(t) = (\theta_0, \varphi_0 + t \sin \theta_0), \quad (38a)$$

$$X_{(\theta_0, \varphi_0)}(t) = (\theta_0 + t, \varphi_0). \quad (38b)$$

So $Xf = \frac{\partial f}{\partial \theta}$ and $Yf = \sin \theta \frac{\partial f}{\partial \varphi}$,

$$(XY)f = \frac{\partial^2 f}{\partial \theta \partial \varphi} \sin \theta + \frac{\partial f}{\partial \varphi} \cos \theta \quad (39)$$

$$(YX)f = \sin \theta \frac{\partial^2 f}{\partial \theta \partial \varphi}.$$