Imaginary diagonal relaxations for highly indefinite linear systems

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Abstract

We investigate preconditioning techniques based on incomplete factorization to solve highly indefinite linear systems. In contrast to the positive definite case, increasing the fill-in level could spoil the convergence of Krylov subspace methods, mainly because of the appearance of eigenvalues around the origin. To overcome this inconvenience, we introduce imaginary diagonal relaxations, which amounts to slightly moving the spectrum of the original system matrix along the imaginary axis. Spectacular improvement is obtained.

Key words: Indefinite linear systems, incomplete factorizations, diagonal relaxations, Krylov subspace methods.

AMS subject classifications: 65F10, 65F15, 65F35, 65F50, 65N22, 65N30, 65E05.

1 Introduction

Let us consider solving highly indefinite large sparse symmetric linear system

(1) $$Az = b \qquad \text{with} \qquad A \in \mathbb{C}^{n\times n} \ , \ b, \ z \in \mathbb{C}^{n} \ ,$$

by Krylov subspace iterative methods [2, 14]. For efficiency purposes, we solve the preconditioned system

(2) $$AB^{-1}\tilde{z} = b \qquad \text{with} \qquad z = B^{-1}\tilde{z}$$

where the precontitioning matrix B is constructed by means of some incomplete factorization of A. The eigenvalues of AB^{-1} move towards 1 as one increases the number of accepted fill-in entries in B. Unfortunately,

- the eigenvalues with negative real part, on their (erratic) way towards 1 have to cross the origin, whence from time to time the appearance of eigenvalues very close to 0, [3, pp. 198-199];
- the "convergence" of eigenvalues towards 1 is not monotonous; a decrease of the real part may coincide with an increase of the imaginary part, and vice versa [10];

which could slow down the convergence, [3, 10, 11]. To get around this bottleneck, two approaches are explored:

1. The real part of the system matrix is made positive definite, or less indefinite, by means of real diagonal relaxations, [10].
2. The system matrix is slightly moved along the imaginary axis by incorporating imaginary diagonal relaxations, [11].

The latter technique turns out to be more efficient. Compared to standard incomplete factorizations, the new preconditionings, which consist in applying standard incomplete factorizations to the matrix A perturbed by a diagonal complex matrix, lead in general to a spectacular decrease of the number of Krylov subspace iterations.

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2 Description of the preconditionings

2.1 Moving along the real axis

Let $\gamma \geq 0$ denote a parameter, and Q stand for the diagonal matrix whose diagonal entries q_{ii} are defined by

(3) $$q_{ii} = -\gamma \, \min \{0 \, , \, Re\,((A\mathbf{e})_i)\} \; .$$

where Re denotes the *real part*, and $\mathbf{e} = (1, \; 1, \cdots, \; 1)^T$. Set

(4) $$A_0 \; = \; Re(A) + Q$$

(5) $$\widetilde{A} \; = \; A_0 + \mathbf{i}\, Im(A) \; = \; A \; + \; Q \; ,$$

where Im stands for the *imaginary part.* Let B_0 and $\widetilde{B}$ denote the preconditioning matrices obtained by applying some (variant of) incomplete factorization technique (see, e.g., [1, 2, 9]) to A_0 and $\widetilde{A}$, respectively, with $1 \lesssim \gamma \lesssim 3$. The following conclusions have been drawn in [10]. B_0 and $\widetilde{B}$ are in general more efficient than the preconditioners obtained by applying incomplete factorization techniques directly to A. Increasing the fill-in level could lead to an increase of the Krylov subspace iteration counts, for reasons as explained in the introduction. Nevertheless, perturbed methods suffer (by far) less than standard methods from the phenomena observed. $\widetilde{B}$ performs in general better than B_0.

Similar diagonal perturbations have been used in [8, 12, 13], but there the main purpose was to ensure the existence of the preconditioner. Here, the emphasis is on clustering eigenvalues of preconditioned systems enough away from the origin.

2.2 Moving along the imaginary axis

Observe that if $Im(A)$ is positive (or negative) semidefinite then $\widetilde{A}$ may be interpreted as a "shift" of A_0 along the imaginary axis. This motivates the introduction of the following auxiliary matrices:

(6) $$\widehat{A}_\xi \; = \; A + \mathbf{i}\,\xi h D_r$$

(7) $$\widetilde{A}_\xi \; = \; A + Q + \mathbf{i}\,\xi h D_r$$

where $D_r = diag(Re(A))$, $\xi \geq 0$ is a parameter, while h denotes the mesh size parameter. In the case of variable mesh size, we suggest to use

(8) $$h = n^{-\frac{1}{d}}$$

where n is the order of the matrix A, while d denotes the space dimension ($d = 2$ or 3).

3 Numerical results

The computations are carried out on a Silicon Graphics Origin 2000 workstation. The zero vector is chosen as initial guess. The residual error reduction $\|r\|/\|b\| \leq 10^{-6}$ is used as convergence criterion. The iteration counts reported correspond to how many times the matrix vector multiplication is executed in the Krylov subspace iterative method involved. The preconditionings include :

1. **IC** : the standard incomplete Cholesky applied to A;
2. **MIC** : the standard modified incomplete Cholesky applied to A;
3. $\widetilde{\mathbf{IC}}$: IC applied to $\widetilde{A}_\xi = A + Q + \mathbf{i}\,\xi h D_r$ $\;(\gamma = 1, \; 2)$;
4. $\widetilde{\mathbf{MIC}}$: MIC applied to $\widetilde{A}_\xi = A + Q + \mathbf{i}\,\xi h D_r$; $(\gamma = 2, \; 3)$;
5. $\widehat{\mathbf{IC}}$: IC applied to $\widehat{A}_\xi = A + \mathbf{i}\,\xi h D_r$;
6. $\widehat{\mathbf{MIC}}$: MIC applied to $\widehat{A}_\xi = A + \mathbf{i}\,\xi h D_r$.

Problem 1

$$
\begin{array}{ll}
-\Delta u - k^2 u = f & \text{in } \Omega = (0,1)\times(0,1) \\
u = 1 & \text{on } \{(0,y)\ ;\ 0 \le y \le 1\} \\
\frac{\partial u}{\partial n} = 0 & \text{on } \{(x,0)\ ;\ 0 \le x \le 1\} \bigcup \{(x,1)\ ;\ 0 \le x \le 1\} \\
\frac{\partial u}{\partial n} + \mathbf{i}ku = 0 & \text{on } \{(1,y)\ ;\ 0 \le y \le 1\}
\end{array}
$$

The Galerkin finite elements method is used with piecewise linear basis functions over isosceles right triangles. Local nodes are ordered counterclockwise starting from the node at the right angle. The lexicographic ordering in the $x - y$ plane is used to define the global node ordering. We use the *generalized minimal residual* method, with restart parameter equal to 30 (GMRES(30)) [15]. The GMRES routine was retrieved from the CERFACS GMRES Package [4]. The results are collected in Table 1 and in Fig. 1.

Table 1: Problem 1 with $k = 20$ and $h^{-1} = 200$ ($n = 40\,200$). Effect of fill level on the number of matrix-vector products, for preconditioned GMRES(30). Symbol $\star$ means stagnation or iteration counts larger than 1000 before reaching the desired accuracy.

fill level / **Precond**	4	8	12	16	20
IC	151	138	599	$\star$	$\star$
$\widehat{IC}$					
$\xi = 0.1$	148	127	739	94	28
$\xi = 0.2$	146	122	339	58	30
$\xi = 0.5$	144	100	79	45	38
$\xi = 0.7$	143	90	57	43	40
$\xi = 0.9$	142	86	53	47	44
$\xi = 1$	141	84	54	48	46
$\widetilde{IC}(\gamma = 1)$					
$\xi = 0$	145	102	89	89	106
$\widetilde{IC}(\gamma = 2)$					
$\xi = 0$	143	93	79	68	70
$\xi = 0.1$	141	91	77	65	67
$\xi = 0.2$	140	90	75	64	64
$\xi = 0.5$	141	86	68	58	55
$\xi = 0.7$	140	84	66	55	53
$\xi = 0.9$	139	85	65	56	52
$\xi = 1$	138	83	62	55	53

fill level / **Precond**	4	8	12	16	20
MIC	$\star$	$\star$	$\star$	$\star$	$\star$
$\widehat{MIC}$					
$\xi = 0.1$	141	69	49	41	38
$\xi = 0.2$	148	70	50	45	43
$\xi = 0.5$	153	84	59	50	47
$\xi = 0.7$	161	86	60	52	49
$\xi = 0.9$	147	87	64	55	50
$\xi = 1$	147	90	66	56	51
$\widetilde{MIC}(\gamma = 2)$					
$\xi = 0$	161	101	88	83	83
$\widetilde{MIC}(\gamma = 3)$					
$\xi = 0$	132	89	82	83	81
$\xi = 0.1$	127	87	81	81	81
$\xi = 0.2$	122	87	81	81	78
$\xi = 0.5$	122	85	78	73	70
$\xi = 0.7$	118	84	76	72	71
$\xi = 0.9$	126	85	77	69	64
$\xi = 1$	126	83	78	74	71

Problem 2 *The data were provided, in modified compressed sparse row (MCSR) format, by J.C. Autrique from LMS International. It consists of an acoustic modal analysis of an exhaust pipe (silencer) of a Jaguar car. The finite element model represented in Fig. 2 involves* 39 254 *hexahedral elements (HEXA8) and* 46 966 *nodes. The matrix* A*, which is complex-symmetric, has* 586 925 *nonzero entries in its lower triangular part. The wave number is* $k = 6\pi$*, whose corresponding frequency is* 1 500 *Hz.*

Now, we also include the *quasi-minimal residual* (QMR) method [5]. Unlike GMRES, QMR does not minimize the true residual but a related vector. As regards storage requirement and computational costs per iteration step, QMR is cheaper than GMRES. We have used the coupled two-term version of QMR for *complex-symmetric* matrices [6].

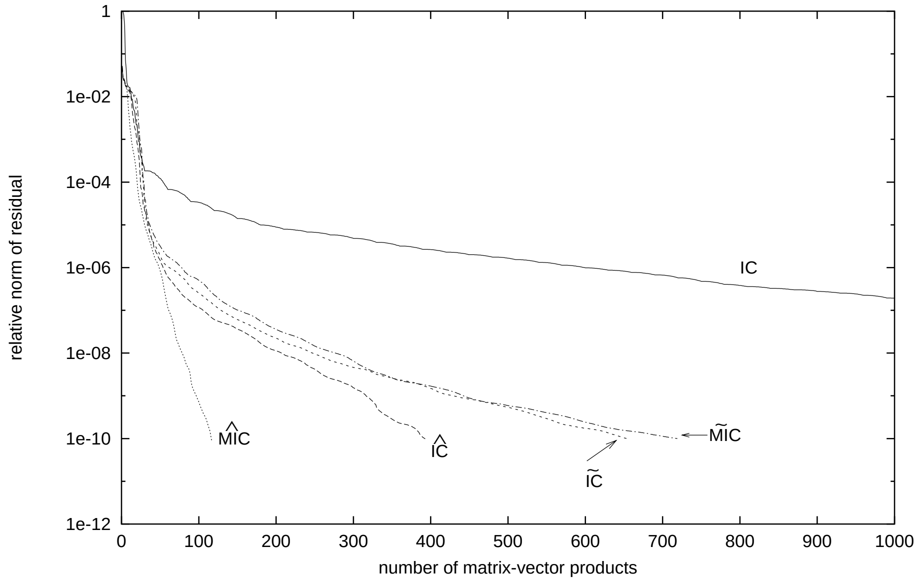

Figure 1: Problem 1 with $h = 1/200$ ($n = 40200$) and $k = 20$. Evolution of the relative residual error for GMRES(30). Fill level $= 12$. $\widehat{IC}$ was used with $\xi = 1$, $\widetilde{IC}$ with $\gamma = 2$ and $\xi = 1$, $\widehat{MIC}$ with $\xi = 0.1$, and $\widetilde{MIC}$ with $\gamma = 3$ and $\xi = 1$.

The routine was retrieved from the QMR package *QMRPACK* [7]. We now solve the (complex-symmetric) two-sided preconditioned system

(9) $$\tilde{A}\tilde{z} = \tilde{b} \quad \text{where} \quad \tilde{A} = P^{\frac{1}{2}}L^{-1}AL^{-t}P^{\frac{1}{2}} \quad \text{and} \quad \tilde{b} = P^{\frac{1}{2}}L^{-1}b \ .$$

The solution to the original system (1) is $z = L^{-t}P^{\frac{1}{2}}\tilde{z}$. In the QMR algorithm, the norm of the (preconditioned) residual is not available, in contrast to GMRES. We have opted for computing the true preconditioned QMR residual at each iteration step. This requires an additional matrix-vector product, which we do not count in Fig. 3, where the evolution of the preconditioned relative residual is reported for IC, $\widehat{IC}$, and $\widehat{MIC}$. In the case of Problem 1 ($h = 1/200$, $n = 40200$), we have used $\ell = 12$, $\xi = 0.9$ for $\widehat{IC}$, and $\xi = 0.05$ for $\widehat{MIC}$. Problem 2 is handled with $\ell = 6$, $\xi = 0.01$ for $\widehat{IC}$, and $\xi = 0.03$ for $\widehat{MIC}$.

The gain in iteration counts obtained by $\widehat{IC}$ and $\widehat{MIC}$ over their standard counterparts is in general considerable, in particular for *modified* methods ($\widehat{MIC}$), whose standard variants (MIC) stagnate or do not convergence at all. The convergence of GMRES appears to be very sensitive to the dimension of the Krylov subspace basis. From all our numerical experiments (see also [10, 11]), the best all-around strategy seems to be the QMR method preconditioned with $\widehat{MIC}$, preferably, or with $\widehat{IC}$.

4 Conclusions

To prevent high level incomplete factorization preconditionings from disappointing performances, appropriate imaginary diagonal relaxations should be added to the matrix involved. This clusters and moves the eigenvalues of the preconditioned system (enough) away from the origin. A spectacular improvement in the convergence rate is achieved over standard methods. It would be of interest to apply our preconditionings to other applications involving strongly indefinite matrices. For instance, in electromagnetics (antenna propagation and radar scattering problems), seismology, and multiple radiation problems.

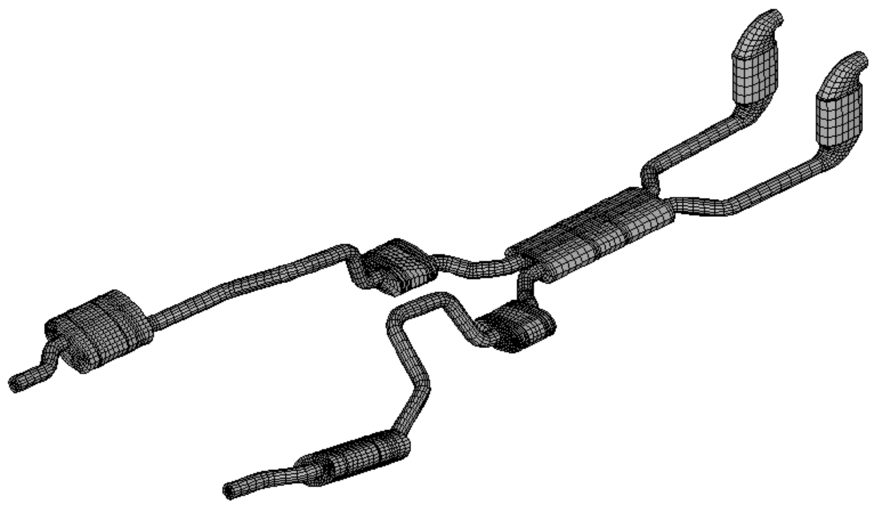

Figure 2: Problem 2. Finite element model for the exhaust pipe.

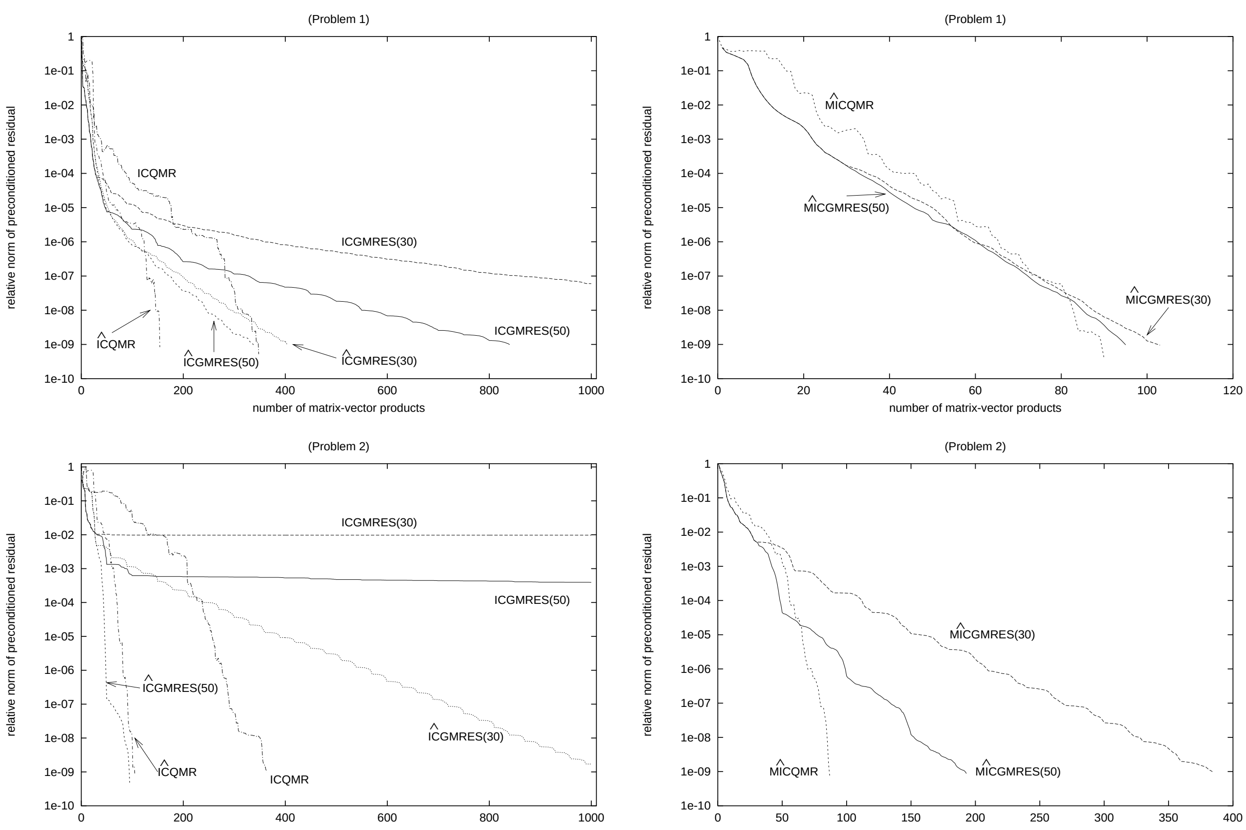

Figure 3: Evolution of the relative preconditioned residual error for GMRES(m), $m = 30, 50$, and for QMR.

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