

Cannibal Animal Games: a new variant of Tic-Tac-Toe

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Abstract

This paper presents a new partial two-player game, called the *cannibal animal game*, which is a variant of Tic-Tac-Toe. The game is played on the infinite grid, where in each round a player chooses and occupies free cells. The first player Alice, who can occupy a cell in each turn, wins if she occupies a set of cells, the union of a subset of which is a translated or rotated copy of a previously agreed upon polyomino P (called an *animal*). The objective of the second player Bob is to prevent Alice from creating her animal by occupying in each round a translated or rotated copy of P . An animal is a *cannibal* if Bob has a winning strategy, and a *non-cannibal* otherwise. This paper presents some new tools, such as the *bounding strategy* and the *punching lemma*, to classify animals into cannibals or non-cannibals. It is also shown that the *pairing strategy* also works for this problem.

1 Introduction

Studying variants of the Tic-Tac-Toe game are interesting problems in the area of recreational mathematics [1, 2, 3, 5, 6, 7, 8, 9]. Probably the most studied among these games is an achievement game, a somewhat generalized Tic-Tac-Toe, presented by Harary [3, 5]. A *polyomino* or an *animal* is a set of connected cells (sharing an edge) of the infinite grid. In such games, two players Alice and Bob alternatively occupy one cell of the infinite grid in each round of the game, and the player who is the first to occupy a translated copy of the given animal is a winner (we will always assume that Alice is the first player). In these games Bob cannot win, hence his objective is to obstruct Alice's achievement.

Here we present a new achievement game called the *cannibal animal game*. As with Harary's achievement game, it is played on the infinite grid whereby players alternate turns to occupy free cells of the grid. This means that in each round the player must choose grid

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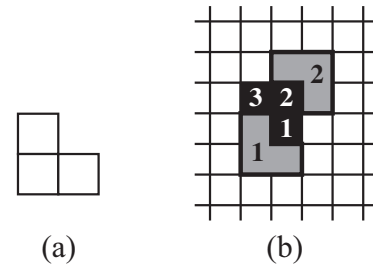


Figure 1: (a) The animal El (an L-shaped triomino), (b) An example of the progress of the game: cells depicted in black are occupied by Alice, and animals depicted in grey correspond to Bob's moves. In both cases, the numbers on the cells represent the order in which the cells are occupied. In the example, Alice wins.

cells that are not yet occupied; hence, occupied regions do not intersect. Moreover, once a cell is occupied, it remains so until the end of the game. In contrast to the generalized Tic-Tac-Toe, the cannibal animal game is a *partial game*: the roles and legal moves of Alice and Bob are different. Alice's legal move is to occupy one cell of the infinite grid in each round, and she wins if she occupies a translated copy of an animal given beforehand (this move is the same as that of the first player of Harary's generalized Tic-Tac-Toe). Bob's role and allowed moves, however, are different: in each round he must occupy a copy of the given animal (hence occupy a subset of the grid cells), and his objective is to prevent Alice from achieving the animal. The animal achieved or that Bob occupies may be a translation, a mirror image and/or a 90, 180, or 270-degree rotation of the given animal. Each such translation/rotation is called a *copy* of the animal and n -cell-animal is an animal consisting of n cells. Figure 1 shows an example of the progress of the game where the animal is El , an L-shaped triomino.

We call an animal a *cannibal* or a *loser* if Bob has a winning strategy (Bob's animal eats Alice's animal) and a *non-cannibal* or a *winner* otherwise. And hence the game is called the *cannibal animal game*.

Our Results. In this paper we study the following animals (see Figure 2 for examples): $R(n, m)$ is an

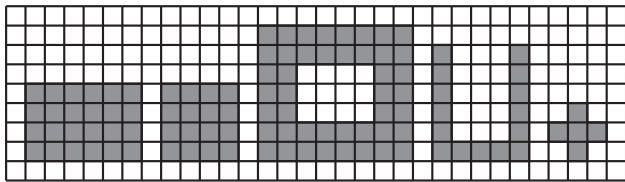


Figure 2: Examples of animals: $R(4,6)$, $S(4)$, $O(7,8,2)$, $U(6,5,1)$, and $X(3)$ (from left to right).

nm -cell-animal of an $n \times m$ rectangle. $R(n,n)$ is sometimes expressed as $S(n)$ (or an $n \times n$ square). $O(n,m,k)$ (for $k < \min\{n/2, m/2\}$) is a $2k(n+m-2k)$ -cell-animal having the shape of $R(n,m)$ but with a hole of $(n-2k) \times (m-2k)$ rectangle in the center (that is, an O -shaped polyomino whose thickness is k). $U(h,w,k)$ (for $k < \min\{h, w/2\}$) is a $k(2h+w+2k)$ -cell-animal having a U -shape with height h , width w , and thickness k . $X(n)$ (or n -cross) is a $(2n-1)$ -cell-animal consisting of vertical and horizontal cells, each of length n , that cross each other at the center cells.

1. The following animals are cannibals:
 - (a) $S(n)$ with holes if at least one of the holes is at least $\lfloor n/4 \rfloor$ cells away from the boundary for $n \geq 4$ (and no hole is on the boundary),
 - (b) $O(n,m,k)$ for $n, m, k \in \mathbb{N}$,
 - (c) $U(h,w,1)$ for $h, w \in \mathbb{N}$, except $U(2,4,1)$.
2. The following animals are non-cannibals:
 - (a) Animals with at most three cells,
 - (b) $R(n,m)$ for any $n, m \in \mathbb{N}$,
 - (c) $X(3)$.

2 Cannibal animals (losers) and pairing strategy

In this section we demonstrate a strategy for Bob that prevents Alice winning for a few families of animals. To do this, we will use the idea of *pairing strategy* that is used in many other combinatorial games. In what follows we will show that this approach also works for our cannibal animal game. We start with a simple strategy for Bob that works for the $O(n,m,k)$ animal:

Theorem 1 $O(n,m,k)$ is a cannibal for any $n, m \geq 3$ and $k < \min\{n/2, m/2\}$.

Proof. Bob virtually partitions the playing-board into blocks of size $(n+k) \times (m+k)$. That is, we define the block B_{ij} as the rectangle $[i(n+k), (i+1)(n+k)-1] \times [j(m+k), (j+1)(m+k)-1]$ (as shown in Figure 3). The strategy for Bob is to place his animal inside the block where Alice played her last move. After Alice plays, Bob checks which block her last move belongs to; if he has already played an

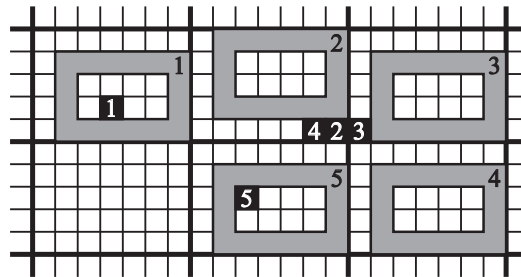


Figure 3: Winning strategy for Bob for $O(n,m,k)$ (in this example, for $O(4,6,1)$). Alice's moves are marked in black and Bob's in grey. The numbers on the cells represent the order in which the cells are occupied. Since the block inside which Alice's 4th move is played already includes Bob's animal, Bob's 4th move is played in another arbitrary block.

animal in that same block, he simply plays in an arbitrary empty block (e.g., Bob's 4th move in Figure 3). Note that since the playing board is infinite, Bob can always play these moves. With this strategy, Alice clearly cannot construct a copy of $O(n,m,k)$. \square

This strategy is also useful for other animals. Recall that by Theorem 6 squares are non-cannibals. Surprisingly, the removal of a single interior cell from a square animal can transform it into a cannibal.

Lemma 2 For any integer $n \geq 4$, let A be an $n \times n$ square animal in which a single cell whose distance to the boundary is at least $\lfloor n/4 \rfloor$ units has been removed. Then A is a cannibal.

Proof. The proof is analogous to the proof of theorem 1. This time we partition the board into blocks of size $(n + \lfloor (n-1)/2 \rfloor) \times (n + \lfloor (n-1)/2 \rfloor)$. If the hole (removed cell) is at least $\lfloor n/4 \rfloor$ units away from the boundary, then Bob can always play his animal inside the same block as Alice's last move (details omitted in this version).

Assume that Alice is able to construct a copy of the animal on the board. By the porthole principle, there would be a block in which Alice's pieces form a square of size at least $\lceil n/2 \rceil \times \lceil n/2 \rceil$ (possibly with one cell removed). However, this cannot occur since Bob also occupies the same block with an $n \times n$ square. \square

In some cases, we might also need a more careful partitioning of the grid into blocks:

Theorem 3 For any $h, w \in \mathbb{N}$ (other than $(h,w) = (2,4)$), the $U(h,w,1)$ animal is cannibal.

Proof. In this case, Bob virtually partitions the playing board into blocks of size $(w+k) \times h$. But if he arranges these blocks naively, there might be "cracks" between Bob's animals in which Alice could construct

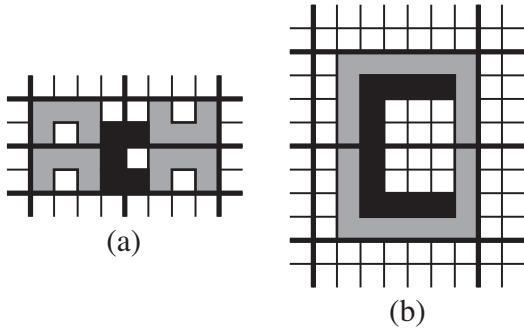
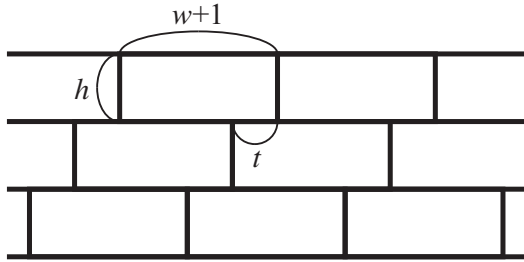


Figure 4: Examples of failed partitions.

Figure 5: Tiling and shift size t .

her animal (see Figure 4). To avoid such cracks, Bob must slant his partition, thus tiling the grid with blocks with a shift of size (distance) t (Figure 5). We define the block $B_{i,j}$ as the rectangle $[i(w+1)+jt, i(w+1)+w+jt] \times [jh, jh+h-1]$. The exact value of the slant depends on the parameters w and h :

$h = 2$ (and $w \neq 4$): $t = 2$.

$h \geq 3$ and $2h - 2 \geq w \geq h - 2$: $t = \lfloor (w+1)/2 \rfloor$.

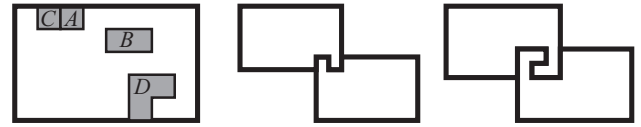
Otherwise: No slant is necessary (i.e., $t = 0$).

It is easy to show that with such a partition, Alice will be unable to construct her animal (details omitted in this version). \square

We now introduce another idea to generate new cannibal animals from known cannibal animals. Let A be an animal and let C be a subset of cells of A . Then $A \setminus C$ is an animal created by removing C from A . We say that C is an *outer piece* if we can locate a second copy of $A \setminus C$ that covers a part of the removed piece C of the first copy; we call C an *inner piece* otherwise. See Figure 6.

Notice that even if C and C' are both inner pieces, $C \cup C'$ may be outer. If C is an outer piece, then for any superset C' of C is also outer.

Lemma 4 (Punching Lemma) *Let A be a cannibal and let C be an inner piece of A . The animal $A \setminus C$ is also a cannibal.*

Figure 6: A and B are inner pieces. C and D are outer pieces since a second copy covers a part of the piece as seen in the right examples.

Proof. Assume otherwise that $A \setminus C$ is a non-cannibal; then Alice would have a winning strategy, i.e., she will be able to construct a copy of $A \setminus C$ without Bob preventing it. Consider now the removed piece C of the animal Alice constructed. Because C is inner, this position cannot be occupied by Bob. Moreover, Alice can occupy this position in another round to form animal A . Thus, a contradiction. \square

Note that the reciprocal is not always true (see for example Theorem 6 and Lemma 2). As a simple application of this lemma, we have the following result:

Theorem 5 *For any integer $n \geq 4$, let S' be an animal $S(n)$ in which any number of interior cells have been removed. If at least one of the removed cells has distance $\lfloor n/4 \rfloor$ or more to the boundary, then S' is a cannibal.*

3 Non-cannibal animals (winners) and the bounding strategy

In this section we give a few families of non-cannibal animals. We first introduce a concept on common intersection:

Definition 1 *An animal P is called 2-Helly [4] if for any family \mathcal{A} of copies of P such that $A \cap A' \neq \emptyset$ for any $A, A' \in \mathcal{A}$, the intersection $\bigcap_{A \in \mathcal{A}} A$ is nonempty.*

It is easy to see that $R(n, m)$ for any n and m is 2-Helly, while none of the other animals that we study are.

Theorem 6 *Any 2-Helly animal is a non-cannibal.*

For proving this theorem, we first prove a restricted version as follows:

Lemma 7 *In any finite board, any 2-Helly animal P is a non-cannibal provided that at least one copy P can be occupied on the board.*

Proof. At the beginning of each round we define $\mathcal{S} = \{s_1, \dots, s_k\}$ as the set of copies of P not occupied by Bob that fit in the board (note that some of these positions may be occupied by Alice's previous moves). The set \mathcal{S} will be treated as a set of potential positions in which Alice may form her animal. Note that Bob's

moves must be at some $s \in \mathcal{S}$. Also, let $\mathcal{S}' \subseteq \mathcal{S}$ be the set of animals that stab all elements of \mathcal{S} (that is, $s' \in \mathcal{S}' \Leftrightarrow s' \cap s \neq \emptyset, \forall s \in \mathcal{S}$).

Note that the set \mathcal{S} initially is nonempty at the beginning of the game, and whenever Bob plays, the size of \mathcal{S} is reduced. However, notice that \mathcal{S} will only become empty if Bob can place his copy occupying the cells of some $s' \in \mathcal{S}'$. Hence, Alice's strategy is as follows: if the set \mathcal{S}' is empty, Alice occupies any empty cell of some $s \in \mathcal{S}$. Otherwise, \mathcal{S}' is a nonempty set of animals where any two intersect. Hence, by 2-Hellyness, there exists a cell c that intersects all the animals of \mathcal{S}' . Alice will occupy c , preventing Bob from occupying any cell of \mathcal{S}' .

With this strategy, Alice makes sure that the set \mathcal{S} never becomes empty (since Bob can never occupy $s' \in \mathcal{S}'$) and the number of Bob's possible moves only decreases after each of Alice's moves. Hence after a finite number of turns, Bob will be unable to play inside the square and Alice will be able to complete a copy of the animal. \square

The result of Lemma 7 can be extended to an infinite board, whereby Alice's strategy is to construct a bounded region big enough so that the set \mathcal{S} is nonempty. If she can construct such a region, she can apply the strategy of Lemma 7 by playing only inside this bounded region: From this idea we have the proof of Theorem 6 as follows:

Proof of Theorem 6 (Bounding strategy). Given P , let n and m be the smallest integers such that P is included in $R(n, m)$ (that is $R(n, m)$ is the smallest rectangle that can enclose P). We will construct an $N \times N$ square region on the board large enough that at least one copy of $R(n, m)$ can be constructed inside (hence so will P). Alice can surround the boundary of the $(N + 2) \times (N + 2)$ square with at most $4N$ moves (note that the four corners don't need to be occupied). Let I be the interior of the square. Notice that at least $2(N - (n - 1))(N - (m - 1))$ copies of $R(n, m)$ can fit inside I . Each of Bob's animals stabs at most $(2n - 1)(2m - 1) + (n + m - 1)^2 \leq n^2 + m^2 + 6nm$ copies of $R(n, m)$.

During the (at most) $4N$ rounds during which Alice surrounds the boundary of the square, Bob can stab at most $4N(n^2 + m^2 + 6nm)$ animals of \mathcal{S} . Thus, if $2(N - n + 1)(N - m + 1) > 4N(n^2 + m^2 + 6nm)$, the set \mathcal{S} will be non-empty even after Alice has completed surrounding the boundary of the square. Because the first term is quadratic in N and the second is linear, for a sufficiently large N the inequality holds. \square

Corollary 8 $R(n, m)$ is a non-cannibal (for any $n, m \in \mathbb{N}$).

For some simple animals, we can construct concrete winning strategies for Alice as follows (For space lim-

itation, their proofs are omitted.):

Lemma 9 For $S(n)$ and $X(3)$, Alice can win by at most $n^2 + 3$ and 8 moves, respectively.

4 Concluding remarks

In Harary's generalized tic-tac-toe, some monotone properties hold; these properties include "increasing the size of the board helps Alice" and "increasing the animal helps Bob." However, such properties do not hold for the cannibal animal game, making it deeper and more interesting. We also note that the cannibal property of many other animals is still left unsolved. Among them is the $U(2, 4, 1)$ animal, which we conjecture to be a cannibal, and the squares $S(n)$ in which a cell less than $\lfloor n/4 \rfloor$ units away from the boundary has been removed. On the other hand, it is easy to see that any animal consisting of at most 3 cells is a non-cannibal. We conjecture that all 4-cell-animals are also non-cannibals, and consequently, the 5-cell-animal $U(2, 3, 1)$ would be the smallest cannibal.

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References

- [1] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways for Your Mathematical Plays*, Vol. 3, Second Edition, A K Peters, Massachusetts, 2003.
- [2] E. Fisher and N. Seiben, Rectangular polyomino set weak (1, 2)-achievement games, *Theoretical Computer Science*, **409**, 2008, pp. 333–340.
- [3] M. Gardner, *Generalized Tictactoe*, in: *Fractal Music, Hypercards and More...*, W. H. Freeman and Company, NY, 1992, pp. 202–213.
- [4] J.E. Goodman and J. O'Rourke, *Handbook of Discrete and Computational Geometry*, CRC Press, 1997.
- [5] F. Harary, Achieving the skinny animal, *Eureka*, **42**, 1982, pp. 8–14. Errata in Edition Number 42, **43**, 1983, pp. 5–6.
- [6] F. Harary and H. Harborth, Achievement and avoidance games with triangular animals, *J. Recreational Mathematics*, **18(2)**, 1985–86, pp. 110–115.
- [7] H. Harborth and M. Seemann, Handicap achievement for squares, *J. Combin. Math. Combin. Comput.*, **46**, 2003, pp. 47–52.
- [8] H. Ito and H. Miyagawa, Snaky is a winner with one handicap, *Abstracts of HERCMA 2007*, Sept. 20–22, Athens Univ. of Economics and Business, Athens, Greece, 2007, pp. 25–26.
- [9] J. Beck, *Combinatorial Games — Tic-Tac-Toe Theory*, Cambridge Univ. Press, 2008.