Evenness Preserving Operations on Musical Rhythms

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ABSTRACT

In this paper we define four operations on musical rhythms that preserve a property called *maximal evenness*. The operations we describe are *shadow*, *complementation*, *concatenation*, and *alternation*.

Categories and Subject Descriptors

F.2 [**Analysis of Algorithms and Problem Complexity**]: Non-numerical Algorithms and Problems—*Geometrical problems and computations*

General Terms

Music Theory

Keywords

music, rhythms, operations, geometry

1. INTRODUCTION

A musical rhythm is a repeating pattern of onsets (sounded pulses) that occur at irregular intervals. These patterns have been studied by mathematicians, as well as musicians, for centuries. Perhaps the first mathematician to have noted a connection between these two fields is Pythagoras (6th century B.C), who observed that the ratio between a plucked string and the frequency of the tone it produces remains constant as the length of the string is varied [4].

Rhythms can be be seen as two-way infinite binary sequences [11], where each bit represents one unit of time called a *pulse*; a 1-bit represents a played note or *onset* and a 0-bit represents a silence (for example, a sixteenth rest). We will use '$\times$' for '1' and '$\cdot$' for '0'. It is generally assumed that the two-way infinite bit sequence is periodic with some period n. A representation that better captures this cyclic nature of rhythms is that of a polygon inscribed in a circle, called a *cyclic polygon*. Consider a circle with n points placed at equal distances around its circumference. A k-sized subset of these points represents a rhythm with n pulses and k onsets. Connecting pairs of points consecutively along the circumference gives us a cyclic polygon.

A musical piece may contain different rhythms and melodies throughout its progression. What musicians often like to do however is to stay within a theme during this progression. For example, a jazz soloist must respect the style and feeling of the piece, and thus play an improvised variation based on the foundation of the main theme [5]. A way of realizing such an improvisation is by taking the base rhythm and transforming it to another through one or more *operations*. An operation transforms one musical rhythm to another based on a set of rules. Operations are important for music composition and especially for improvisations. Thus, studying operations is useful both for theoretical analysis as well as providing formal rules for improvisation techniques. Here we study four operations that preserve a property of rhythms called maximal evenness. The operations we study are: *shadow*, *complementation*, *concatenation*, and *alternation*.

2. DEFINITION AND NOTATION

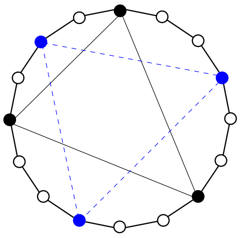
Let R be a rhythm with n pulses and k onsets. When R is represented as a cyclic polygon, the *Euclidean distance* between two onsets i and j is the length of the chord determined by i and j. We say that R is *maximally even* if the sum of all pairwise Euclidean distances between onsets on the circle is maximized. We distinguish between two rhythms R and R' that differ by a rotation of the pulses and onsets. A *Euclidean rhythm* $E(k, n)$ with n pulses and k onsets is a maximally even rhythm that can be generated by the $EUCLIDEAN$ algorithm [3] that we reproduce below:

Algorithm $\text{EUCLIDEAN}(k, n)$

1. **if** k evenly divides n
   then return $(\underbrace{\frac{n}{k}, \frac{n}{k}, \ldots, \frac{n}{k}}_{k})$
2. $a \leftarrow n \bmod k$
3. $(x_1, x_2, \ldots, x_a) \leftarrow \text{EUCLIDEAN}(k, a)$
4. **return** $(\underbrace{\lfloor \frac{n}{k} \rfloor, \ldots, \lfloor \frac{n}{k} \rfloor}_{x_1 - 1}, \lceil \frac{n}{k} \rceil;$
   $\underbrace{\lfloor \frac{n}{k} \rfloor, \ldots, \lfloor \frac{n}{k} \rfloor}_{x_2 - 1}, \lceil \frac{n}{k} \rceil; \ldots;$
   $\underbrace{\lfloor \frac{n}{k} \rfloor, \ldots, \lfloor \frac{n}{k} \rfloor}_{x_a - 1}, \lceil \frac{n}{k} \rceil)$

2.1 Shadow

Several ethnomusicologists have argued that African drumming, handclapping, and mallet performance is best understood as a motor activity. For example a hand (arm) is raised and then dropped



1: The Cuban tresillo and its shadow.

to strike the instrument. According to Jay Rahn [9], one possible mechanism for the tacit motor mediation of attack points of onsets is the peaking of the gesture at the temporal midpoint between two sounds. He calls the sequence of midpoints of the onsets of a rhythm the *shadow* of the rhythm. For example the Cuban *tresillo* rhythm given by [× · · · · · × · · · · · · × · · ·] has the shadow [· · · × · · · · · · × · · · · × ·], which yields the shadow rhythm [× · · · · · × · · · · · × · · · ·]. See Figure 1.

Performed on a rhythm, the shadow operation increases the evenness of the new rhythm. One question that comes to mind is: what happens to the sequence obtained when we continue to perform this operation on every rhythm resulting from the shadow of another? It turns out that this is a geometric problem that has been investigated in the mathematics literature.

The study of properties of sequences of polygons generated by performing iterative processes on an initial polygon P^0 has received much attention, and the shadow operation is one of many operations that has been investigated. Perhaps the most studied sequence is the one sometimes referred to as Kasner polygons [10]. Given a polygon P^0, the Kasner descendant P^1 of P^0 is obtained by placing the vertices of P^1 at the midpoints of the edges of P^0 (making P^1 the shadow of P^0).

Let $P^0, P^1, P^2, \ldots$ be a sequence of cyclic polygons such that the vertex set of P^i is the shadow of the vertex set of P^{i-1} for all integers $i \geq 0$. We say that P^i is the shadow polygon of P^{i-1}. Hitt and Zhang [7] show that given any convex cyclic polygon P^0, its shadow sequence converges to a regular polygon. From their proof, it follows that the area of each P^i is greater than or equal to the area of P^{i-1} for any $i > 0$, with equality only when P^i is regular. In their proof of the convergence of the shadow sequence, Hitt and Zhang make use of doubly stochastic matrices and Schur-convex functions. Below we provide a simpler proof that uses a different approach and is more intuitive. We then show a bound on the rate of the convergence of the shadow sequence.

THEOREM 1. *The shadow sequence of any cyclic polygon converges to a regular polygon.*

PROOF. Let P be a cyclic polygon with n intervals (edges). If the range of the lengths of the intervals of P is zero, then the polygon is regular. Suppose the range in not zero. Then after every shadow operation, the longest interval is averaged with one smaller than the longest. Also the shortest interval is averaged with something greater. Thus after at most n shadow operations, the range strictly decreases. Therefore the range is a monotonically decreasing function of the number of shadow operations. It is also bounded from below by zero. Therefore the range has a limit and it is zero, and since the average value of the intervals is $1/n$ throughout this process, it follows that the limit polygon of P is regular. □

THEOREM 2. *The shadow sequence of an cyclic polygon converges to a regular polygon in such a way that the variance of the intervals decreases at each step by at least one half.*

PROOF. Let P be an n-gon inscribed in a unit circle, and let $\langle a_0, a_1, \ldots, a_{n-1}\rangle$ be the sequence of interval lengths (arclengths) between two consecutive vertices of P in the clockwise direction. Let $P^{(t)}$ denote the polygon P after t shadow operations, and $a_i^{(t)}$ for $i = 0, 1, \ldots, n-1$ denote its corresponding interval lengths. At any step t, we can write the interval lengths of $P^{(t+1)}$ as the sequence: $\left\{\frac{a_i^{(t)}+a_{i+1}^{(t+1)}}{2} : i = 0, 1, \ldots, n-1\right\}$. Also, the average interval length of $P^{(t)}$ is $1/n$ for any t. Since the interval lengths sum to 1 at any step, we can treat the sequence of intervals as a random variable and compute its variance. We will show that the variance $V^{(t+1)}$ of the sequence of interval lengths at time $t+1$ decreases by a constant fraction of the variance at time t. For simplicity, we will assume $a_i^{(t)} = a_i$; thus,

$$
\begin{aligned}
V^{(t+1)} &= \frac{1}{n}\sum_{i=0}^{n-1}\left(a_i^{(t+1)}\right)^2 - \frac{1}{n^2} \\
&= \frac{1}{n}\sum_{i=0}^{n-1}\left(\frac{a_i + a_{i+1}}{2}\right)^2 - \frac{1}{n^2} \\
&= \frac{1}{n}\sum_{i=0}^{n-1}\frac{a_i^2 + a_{i+1}^2 + 2a_i a_{i+1}}{4} - \frac{1}{n^2} \\
&= \frac{1}{n}\sum_{i=0}^{n-1}\frac{a_i^2 + a_{i+1}^2}{4} + \frac{1}{n}\sum_{i=0}^{n-1}\frac{a_i a_{i+1}}{2} - \frac{1}{n^2} \\
&= \frac{1}{2n}\sum_{i=0}^{n-1}a_i^2 + \frac{1}{2n}\sum_{i=1}^{n}a_i a_{i+1} - \frac{1}{n^2} \\
&= \frac{1}{2n}\sum_{i=0}^{n-1}a_i^2 - \frac{1}{2n^2} + \frac{1}{2n}\sum_{i=0}^{n-1}a_i a_{i+1} - \frac{1}{2n^2} \\
&= \frac{1}{2}\left(\frac{1}{n}\sum_{i=0}^{n-1}a_i^2 - \frac{1}{n^2}\right) + \frac{1}{2n}\sum_{i=0}^{n-1}a_i a_{i+1} - \frac{1}{2n^2} \\
&= \frac{1}{2}V^{(t)} + \frac{1}{2n}\sum_{i=0}^{n-1}a_i a_{i+1} - \frac{1}{2n^2}.
\end{aligned}
$$

To show that V^{t+1} is at most a fraction of $V^{(t)}$ at any step t, we find the maximum value of $\sum_{i=0}^{n-1} a_i a_{i+1}$ subject to the constraint $\sum_{i=0}^{n-1} a_i = 1$. Using Lagrange multipliers it can be easily determined that the maximum value of the above sum is attained when all the a_i's have the same value, that is, when they are all equal to $1/n$. To find this maximal point, we solve:

$$\frac{\partial}{\partial a_j}\left(\sum_{i=0}^{n-1} a_i a_{i+1}\right) + \lambda\left(\sum_{i=0}^{n-1} a_i - 1\right) = 0$$

Differentiating these n equations (for $j = 0, 1, \ldots, n-1$) we get,

$$a_{i-1} + a_{i+1} + \lambda = 0.$$

Using the constraint $\sum_{i=0}^{n-1} a_i = 1$ we find that $a_i = \frac{1}{n}$. Thus, we have:

$$
\begin{aligned}
V^{(t+1)} &= \frac{1}{2}V^{(t)} + \frac{1}{2n}\sum_{i=0}^{n-1}a_i a_{i+1} - \frac{1}{2n^2} \\
&< \frac{1}{2}V^{(t)} + \frac{1}{2n}\sum_{i=0}^{n-1}\frac{1}{n}\cdot\frac{1}{n} - \frac{1}{2n^2} = \frac{1}{2}V^{(t)}
\end{aligned}
$$

Therefore, after every shadow step the variance is at least halved; and since the variance is always bounded below by zero, then it converges to zero as t goes to infinity. This in turn implies that every interval length converges to the mean value, which is $1/n$. Thus, the shadow sequence of any cyclic polygon converges to the regular polygon in such a way that the variance decreases by at least one half at every step. □

On a circular lattice, a vertex a_i of the shadow might not lie on a lattice point (a pulse) thus violating the definition of a rhythm. To avoid this problem, in such a case we move a_i to the nearest lattice point in the clockwise direction. The result is the *discrete shadow* of R. We now can show the following:

THEOREM 3. *The discrete shadow of a Euclidean rhythm R is a rotation of R.*

2.2 Complement

The *complement* of a subset R of a set of elements is the set of all elements that are not in R. Consequently, the complement of a rhythm R with k onsets is the rhythm having as onsets the $n-k$ pulses that are not onsets in R. The study of the complementary sets of sets of intervals in the context of pitch (scales and chords) has received a lot of attention in music theory [8]. Rhythm, on the other hand, has been explored little from its complementarity aspects. Consider the Cuban *cinquillo* rhythm given by [× · ×× · ×× ·]. Its complementary rhythm is [· × · · × · · ×], which is a rotation of the famous Cuban *tresillo* rhythm given by [× · · × · · × ·] (Figure 2). Complementary sets have many applications such as the composition of rhythmic complementary canons [12]. Clough and Douthett [2] show that the complement of a Euclidean rhythm is Euclidean up to rotation. Here we give a simpler proof.

THEOREM 4. *The complement of a Euclidean rhythm is Euclidean up to rotation.*

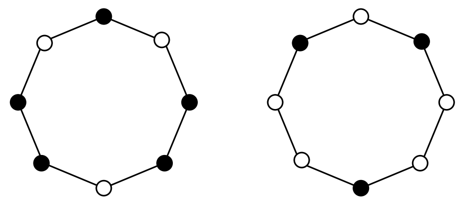
PROOF. Harris and Reingold [6] showed that Euclid's algorithm for computing the greatest common divisor of two integers generates digital straight lines described by the Bresenham algorithm [1]. Euclid's algorithm also generates Euclidean rhythms [3]; thus, a sequence of 0-1 bits of a Euclidean rhythm corresponds to the sequence of 0-1 bits of a digital straight line (modulo rotation). Without loss of generality, assume that a 0-bit of a digital straight line corresponds to a vertical segment and a 1-bit corresponds to a horizontal segment.

Let R be a Euclidean rhythm corresponding to the digital line L defined by the equation $y = ax$ (we assume the line passes through the center of our coordinate system). If we rotate L by $90°$, then the equation of the rotated line L' becomes $x = ay$. To draw the digital line L', we can merely interchange the x and y axis and plot line $L : y = ax$. This means that L and L' are the same digital line and hence are both described by Euclid's algorithm. However, when we rotate L, the 0-bits of R become 1-bits and the 1-bits become 0-bits. Hence we get the complement of R. Since both R and its complement correspond to the "same" digital line drawn by the same sequence of vertical and horizontal segments, they are both Euclidean. □

2.3 Concatenation

The *concatenation* of two rhythms R_1 and R_2, denoted $R_1 \oplus R_2$, is a new rhythm R formed by the pulses of R_1 followed by those of R_2. For example, $E(3,7) \oplus E(4,6) =$ [× · × · × · ·] ⊕ [× · ×× · ×] = [× · × · × · · × · ×× · ×].

THEOREM 5. *For any Euclidean rhythm $E(k,n)$ with $1 \leq k \leq n$, and any natural number c, $E(ck,cn) \oplus E(k,n)$ is a rotation of $E((c+1)k,(c+1)n)$.*



(a) **(b)**

2: (a) The Cuban cinquillo (b) The complement of the Cuban cinquillo is a rotation of the Cuban tresillo.

In general it is not true that the concatenation of two arbitrary Euclidean rhythms is Euclidean. The concatenation of rhythms $E(3,7) \oplus E(4,6)$ is [× · × · × · · × · ×× · ×], which is not equal to $E(7,13)$ or to any of its rotations.

2.4 Alternation

The *alternation* operation transforms every other onset of a rhythm R into a silence. Every rhythm has two alternations: an *even alternation*, where we keep the first onset, and change the second into a silence; and an *odd alternation*, where we keep the second onset, and change the first into a silence. We can show the following:

THEOREM 6. *The even and odd alternations of the Euclidean rhythm $E(2k,2n)$ are rotations of $E(k,2n)$ for any $1 \leq k \leq n$.*

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