A Pumping Lemma for Homometric Rhythms

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Abstract

Homometric rhythms/chords (henceforth: rhythms) are those with the same histogram or multiset of intervals/distances. The purpose of this note is threefold. First to point out the potential importance of *isospectral vertices* in a pair of homometric rhythms. Second, to prove a "pumping lemma" that generates an infinite sequence of homometric rhythms that include isospectral vertices. And finally, to introduce the notion of *polyphonic homometric rhythms*, which apparently have not been previously explored.

1 Introduction

We are concerned with cyclic musical rhythms such as African *timelines*, Indian *talas*, or rhythmic *ostinatos* in Western parlance: these are rhythms that repeat throughout most, if not all, of a piece of music. Such rhythms are conveniently represented on a circular lattice consisting of n evenly spaced points on a circle [?]. These n points are considered as metronomic *pulses*, k of which are sounded (with a percussion instrument), while the remaining $n-k$ are silent. The sounded pulses are termed *onsets*, and may also denote the onset (or start) of a continuous note produced by an instrument such as an organ or trumpet. Alternately, the k onsets (points) may be considered as k pitches making up a musical chord or scale selected from a universe of n pitches [?]. Such sets of points on a circle are called *cyclotomic* sets in the crystallography literature [?], [?]. Every pair of these points determines an inter-onset-duration-interval (the geodesic distance between the pair of points on the circle) [?]. The histogram of this multiset of distances in the context of musical scales and chords is called its *interval content* [?]. Obviously, two rhythms which are congruent to each other have the same interval content (histograms). However, the reason such histograms have received so much attention is that two rhythms with the same histograms need not be congruent. Two sets of points which are not congruent but possess the same multiset of distances are said to be *homometric*, a term introduced by Lindo Patterson in 1939 [?], who first discovered them. Figure 1 shows a pair of incongruent homometric (n,k)=$(12,5)$ rhythms. One of the fundamental theorems in this area is the so-called *hexachordal-theorem* which states that two non-congruent complementary sets with $k = n/2$ (and n even) are homometric [?]. The earliest proof of this theorem in the music literature appears to be due to Milton Babbitt and David Lewin [?]. A much simpler and elegant elementary proof was later found independently by Iglesias in the crystallography literature [?].

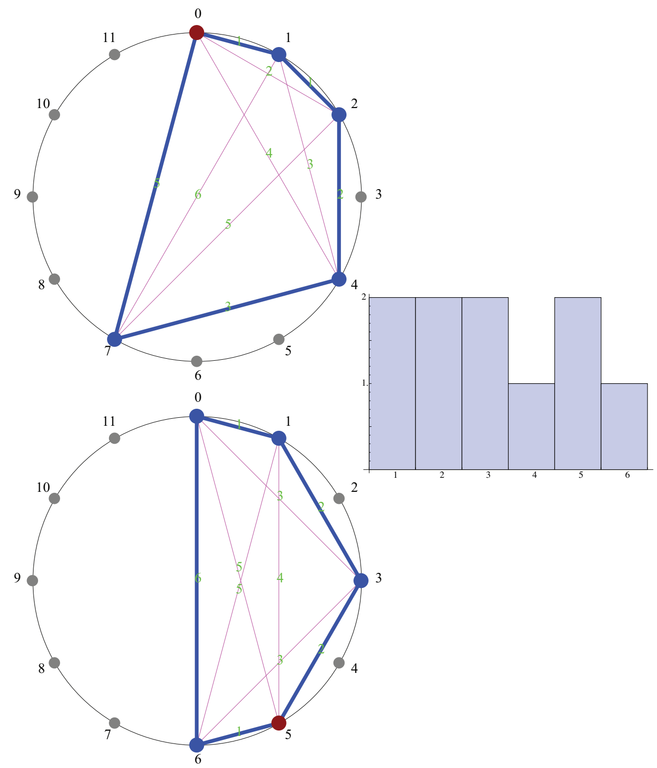

Figure 1: Homometric (n,k)=$(12,5)$ rhythms: $(0,1,2,4,7)$ and $(0,1,3,5,6)$. Vertices 0 and 5 in the first and second rhythms (respectively) are isospectral.

[Mention hexachordal thm?] [Some known xxx
facts about homometric rhythms] xxx

2 Isospectral Vertices

Let P and Q be two different rhythms, with $p \in P$ and $q \in Q$ vertices in each. The vertices p and q are called *isospectral* if they have the same histogram of distances

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to all other vertices in their respective rhythms. In Figure 1, vertex 0 in the first rhythm, and 5 in the second, are isospectral, with spectrum $\{1, 2, 4, 5\}$.

There are two reasons we consider isospectral vertices of potential significance. The first is that removal of a pair of isospectral vertices from a pair of (n, k) homometric rhythms leaves a homometric pair of $(n, k-1)$ rhythms. This raises the possibility of *shelling* a pair of homometric rhythms one beat at a time while retaining homometricity.

Shellings of rhythms play an important role in musical improvisation. For example, most African drumming music consists of rhythms operating on three different strata: the unvarying timeline usually provided by one or more bells, one or more rhythmic motifs played on drums, and an improvised solo (played by the lead drummer) riding on the other rhythmic structures. Shellings of rhythms are relevant to the improvisation of solo drumming in the context of such a rhythmic background. The solo improvisation must respect the style and feeling of the piece which is usually determined by the timeline. A common technique to achieve this effect is to "borrow" notes from the timeline, and to alternate between playing subsets of notes from the timeline and from other rhythms that interlock with the timeline [**?**], [**?**]. In the words of Kofi Agawu [Aga86], it takes a fair amount of expertise to create an effective improvisation that is at the same time stylistically coherent. The borrowing of notes from the timeline may be regarded as a fulfillment of the requirements of style coherence.

Second, as we show in Lemma 1 below, the presence of an isospectral pair permits "pumping" the rhythms to homometric pairs based on a larger $n' > n$. So isospectral pairs serve as a natural "pivot" from which to generate new homometric pairs from old both by removing or adding beats.

This naturally raises the question of whether every homometric pair of rhythms must contain an isospectral pair of vertices. The answer is NO, as illustrated in Figure 2.[1] We leave further investigation of isospectral vertices and shellings to future work.

3 The Pumping Lemma

We define an (m, r)-*pumping* of a pair of rhythms P and Q on $\mathbb{Z}_n$, $m \geq 1$, $r \geq 0$, with isospectral vertices $p \in P$ and $q \in Q$, as a new pair of (n', k') rhythms P' and Q' on $\mathbb{Z}_{n'}$, with $n' = mn$ and $k' = k + 2r$, where p is replaced in P' with $p + \{0, \pm 1, \pm 2, \ldots, \pm r\}$, and similarly q replaced in Q' with $q + \{0, \pm 1, \pm 2, \ldots, \pm r\}$.

Figure 3 shows a (m, r)=$(3, 2)$-pumping of the homometric pair from Figure 1 based on the isospectral pair $p = 0$ and $q = 5$. The original (n, k)=$(12, 5)$ rhythms have been pumped to (n', k')=$(36, 9)$ rhythms. The

[1] This is surely not the smallest example.

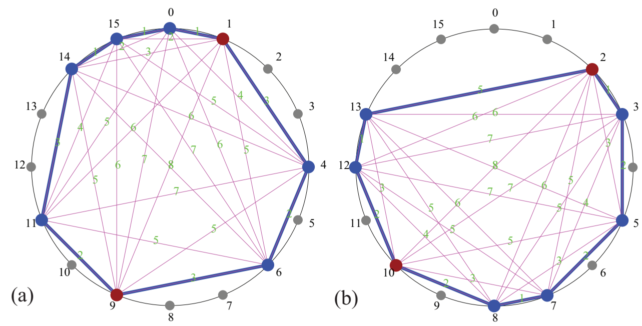

Figure 2: A pair of (n, k)=$(16, 9)$ homometric rhythms that has no isospectral pair of vertices, but does have a pair of two-vertex sets that are isospectral, $\{1, 9\}$ in (a) and $\{2, 10\}$ in (b).

"pumping" occurs both in $n \to mn$ and in $k \to k + 2r$, although it may be that $m = 1$ in which case $n' = n$, or $r = 0$ in which case $k' = k$.

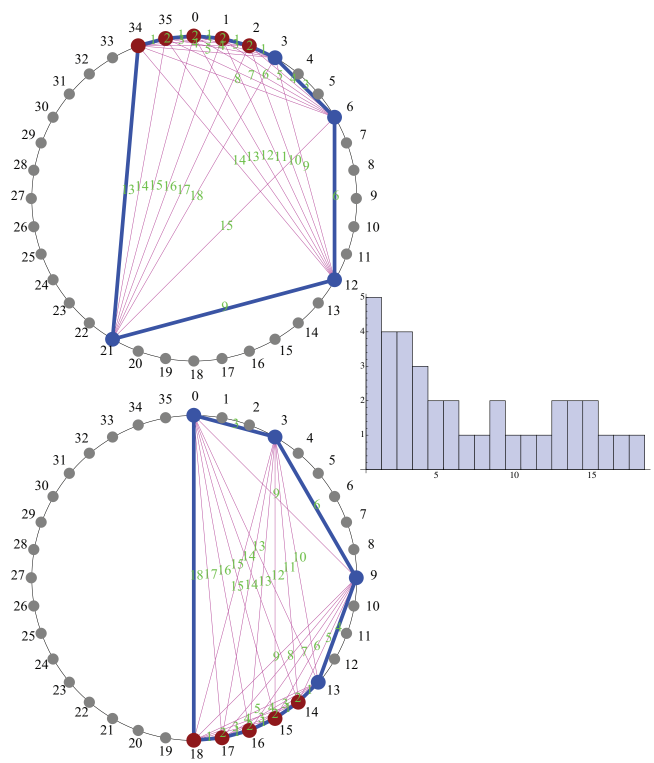

Figure 3: Fig. 1 pumped with $m = 3$, $r = 2$, $n' = mn = 36$. The rhythm is monophonic.

The literature focuses on *monophonic* rhythms, those whose vertices form a set with no repeated elements. The pumping lemma can produce *polyphonic* rhythms, ones in which at least one vertex has multiplicity greater than 1, i.e., the onsets form a multiset. These will be discussed further in Section 4.

Lemma 1 (Pumping) *Let P and Q be a homometric* *pair of (n, k) monophonic rhythms, with isospectral vertices $p \in P$ and $q \in Q$. Then any (m, r)-pumping of P and Q creates a new homometric pair P' and Q'. If $m \geq r + 1$, then the new rhythms are monophonic; if $m \leq r$, the new rhythms could be polyphonic.*

Proof. Call the vertices $p + \{0, \pm 1, \pm 2, \ldots, \pm r\}$ in P'

$$p'_{-r}, \ldots, p'_{-2}, p'_{-1}, p'_0, p'_1, p'_2, \ldots, p'_r$$

and similarly for the q replacements in Q'.

To prove that P' and Q' are homometric, let (x', y') be a segment between two vertices of P'. Consider two cases.

1. Neither x' nor y' is among the p'_i. Then $d(x', y') = md(x, y)$, where x and y are the corresponding vertices in P.

2. $y' = p'_i$. Here there are two subcases. Let $d(x, y) = d(x, p) = d$. Note that the diameter of the circle $\mathbb{Z}_n$ is $n/2$.

   (a) $d = n/2$; or $r \leq n/2 - d$. (Figure 4(a)). Consider the latter inequality. It means that $d \pm r$ does not extend beyond the diameter $n/2$, so that the $p'_{\pm i}$ points and x' all fit inside a semicircle, as in (a) of the figure. Then $d(x', p'_{\pm i}) = md \pm i$ or $md \mp i$, depending on whether the path $x \to p$ or $p \to x$ is shorter, respectively. So, what was the distance d in P between x and $y = p$ becomes the distance set $\{md - r, \ldots, md - 1, md, md + 1, \ldots, md + r\}$ in P'. If d is the diameter $n/2$, then the distance set is $\{md, md - 1, md - 1, md - 2, md - 2, ..., md - r, md - r\}$.

   (b) $r > n/2 - d$ (Figure 4(b)). Here $d \pm r$ does extend beyond the diameter $n/2$, at which point its increase or decrease reverses direction. In (b) of the figure, the new distance set is $\{md - 2, md - 1, md, md + 1, md\}$.

3. Both x' and y' are among the p'_i. Here we get a clique of new distances among the p'_i.

In any of these three cases, call the new distances set $D = \{d(x', p'_{\pm i}) \; : \; i = 0, \ldots, r\}$.

So now we see, in the $P \to P'$ transition, either the change $d \to md$ or $d \to D$. But we see exactly the same distance changes in the $Q \to Q'$ transition. For the distances not involving q are stretched by m, and the distances involving q'_i get stretched by m, $\pm i$, $i = 0, \ldots, r$. Because P and Q are homometric, all the former changes are identical between them, and because p and q are isospectral, all the latter changes are identical between them. Even in the case where the inflation of (x, p) crosses a diameter in the $P \to P'$ inflation, there is a point $z \in Q$ that achieves the same distance

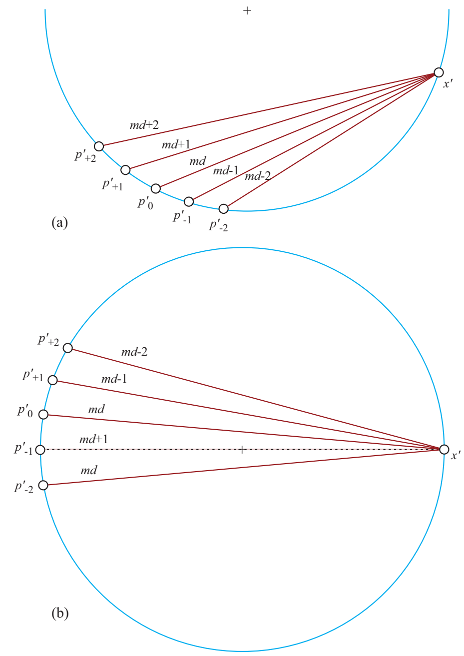

Figure 4: (a) The inflation fits inside a semicircle: $r = 2$, new distances $\{md - r, \ldots, md - 1, md, md + 1, \ldots, md + r\}$. (b) The inflation crosses a diameter, here at (x', p'_{-1}).

$d(x, p) = d(z, q)$ (because p and q are isospectral), so the crossing-diameter behavior, and the distance set D, is exactly mirrored in the $Q \to Q'$ transition.[2] Therefore, P' and Q' are homometric.

We turn now to the mono- and polyphonic claims of the lemma. It should be clear that if we inflate by $m \geq r + 1$, then the closest vertices, separated by 1 in P, become separated by $\geq r + 1$ in P', which is enough to accommodate the addition of r new vertices to one side of p. (If the closest vertices are separated by more than 1, then even smaller inflation will avoid overlap.) Continuing our example, inflation by $m = r + 1 = 3$ suffices to avoid overlap and so maintain a monophonic rhythm, as illustrated in Figure 3.

When $m \leq r$, there could be overlap of the newly added vertices on top of the old vertices. So the resulting rhythm may be polyphonic. However, the rhythms are still homometric, where we treat vertices with multiplicity more than 1 as if they were distinct vertices (and distance 0 is ignored). This is illustrated in Figure 5, where we have used $m = 1$, i.e., $n' = n$. □

The inspiration for the transformation described in this lemma is Property 7 in [?], which similarly inflates a particular pair of homometric quadrilaterals by replacing a vertex in each by a sequence of vertices, and

[2] An instance of this behavior is illustrated in Figure 5 below, where the set D for segments $(7, 0) \in P$ and $(0, 5) \in Q$ have inflated distance set $D = \{3, 4, 5, 6, 5\}$.

increasing n to accommodate. However, their inflation does not rely on what we call "isospectral vertices," and appears to only work on that specific quadrilateral pair.

Corollary 2 *From any pair of rhythms satisfying the preconditions of the pumping lemma, we can generate an infinite sequence of increasingly large homometric pairs.*

The reason for this is that, if p and q are isospectral in P and Q, then p'_0 and q'_0 are isospectral in P' and Q', because their distance spectra are simply scaled by m (most clearly seen in Figure 3). Thus, a pumped pair can be pumped again.

4 Polyphonic Rhythms

As mentioned in the proof above, if we do not pump n enough to accommodate the pumping of k without overlap, i.e., when $m \leq r$, an (m, r)-pumping may convert a monophonic rhythm into a polyphonic rhythm. In general, vertices of a rhythm have integer weights representing their multiplicity. In Figure 5, two vertices have weight 2 whereas all other have weight 1. The interval histogram still makes sense. For example, in the first rhythm's histogram, the distance 6 is achieved three times: by $(10, 4)$ and twice by $(7, 1)$ because vertex 1 has weight 2. And the pumping lemma still guarantees homometricity.

One could interpret onsets of weight greater than 1 as representing greater emphasis, or several different sounding drums, or several voices in the pitch model, where each is an octave apart from the others. Homometricity in polyphonic rhythms is an apparently unexplored topic. See [**?**] for a start at investigations. In the pitch model several voices sounded in unison form the basis of harmony. We believe that polyphonic homometricity will spearhead new directions for research in music theory.

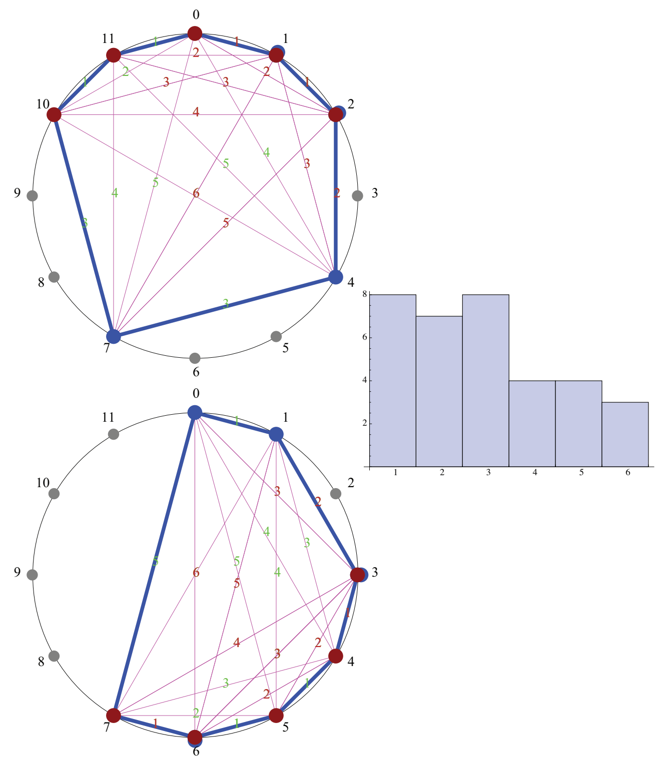

Figure 5: Fig. 1 pumped with $m = 1$, $r = 2$, $n' = n = 12$. The rhythm is polyphonic: $\{1, 2\}$ in the first rhythm, and $\{3, 6\}$, in the second, have multiplicity 2.

References