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Research Article

Interlocking and Euclidean Rhythms

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In this paper we define several operations on Euclidean rhythms based on musical motivations. The operations we define are complementation, alternation, and decomposition. We prove some mathematical properties for each and examine the conditions under which a given operation preserves the Euclidean property. Finally, we show connections to interlocking Euclidean rhythms and tiling canons, and introduce tiling quasi-canons.

Keywords: Euclidean rhythms; interlocking rhythms; operations; complement; alternation; decomposition;

MCS/CCS/AMS Classification/CR Category numbers: 11A05; 52C99; 51K99;

1. Introduction

In this paper we discuss mathematical properties of certain operations applied to a family of musical rhythms called *Euclidean rhythms*, and show connections between these mathematical ideas and interlocking rhythms. We begin by giving an overview of interlocking and Euclidean rhythms. We then proceed to study operations on Euclidean rhythms such as complementation, alternation, and decomposition. We finally relate these mathematical properties to interlocking rhythms.

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1.0.0.1. Interlocking Rhythms. In many musical traditions, the concept of interlocking rhythms is intuitive more than well defined. Although some rhythms are qualified as interlocking in certain contexts, the label is applied almost intuitively. Nonetheless, we see interlocking rhythms in many musical traditions. For example, in Afro-Cuban music interlocking rhythms are found in styles such as the *son* or the *guaracha* where the timbal plays an interlocking rhythm called *cáscara* [15]. That rhythm, shown in Figure 1, played on the sides of the timbal, creates differences in texture. In percussion, especially in funk and jazz, playing the complement of a rhythm very quietly, more felt than heard, is common. This is referred to as playing the ghost notes. Interlocking rhythms are also related to complementary rhythmic canons. Rhythmic canons were first introduced by the French composer Olivier Messian as part of his compositional ideas. A rhythmic canon is a rhythmic pattern played several times, each at a different entry time; it is called complementary when, on each pulse, no more than one pattern has an onset. Rhythmic canons were first studied by Vuza [19] and later by Andreatta [1]. Hall [10] studied the relationship between asymmetric rhythms and tiling canons (tiling canons are defined as complementary canons whose union tiles the whole timespan).

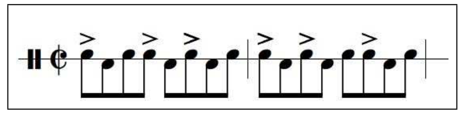

Figure 1. *Cáscara* rhythm.

In Western classical music interlocking rhythms can also be found, although less frequently. For example, in Tchaikovsky's *Romeo and Juliet* Fantasy-Overture, interlocking rhythms are found in the development section – between measures 333-342 the trombone plays a solo melody, while the rest of the orchestra plays on the complement of its onsets. Figure 2 shows four measures of this excerpt in a reduction of the orchestral score.

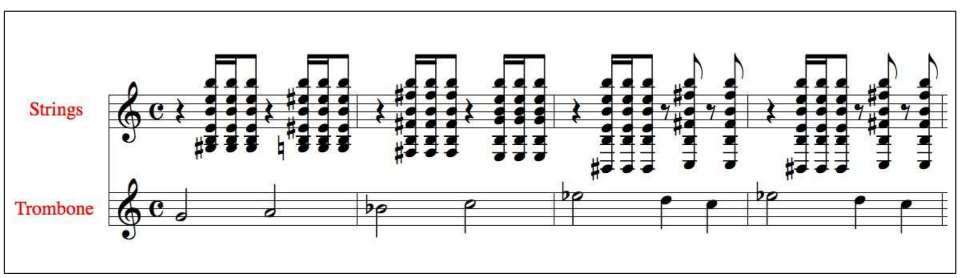

Figure 2. Measures 333-336 from the overture-fantasy *Romeo and Juliet*.

Interlocking rhythms also occur in African music. Anku [2] studied the integration of different parts in African drumming and used musical examples from Akan funeral music to illustrate his theories. He distinguished three possible integration relationships: *overlapping*, *interlocking*, and *adjacency and alternation*. Interlocking rhythms were defined as rhythms that are superimposed on each other so that they share the same orientation as the regulative beat. This definition seems to be incomplete. In his later work Anku's analysis of musical examples gave a sense of musical integration and finality that surpassed his initial definition of interlocking rhythm.

In his book [12] Locke explored the musical structure of *gahu*, a traditional music and dance of the Ewe people of Ghana. The basic polyrhythm of this music is held by three drums, *sogo*, *kidi*, and *kaganu*, from low-pitched to high-pitched, along with the *gankogui* (an iron bell that keeps the timeline). The rhythms played by those three drums result in an interlocking rhythm. As described by Locke, "Sogo's low-pitched bounce starts, kidi's four middle-pitched bounces follow, and kaganu's two high-pitched strokes finish" (see [12], pages 46-47). Figure 3 shows the interlocked parts of the three drums, and the timeline of the *gankogui*; the notation, including beaming, has been borrowed from Locke. In this case, texture reinforces the interlocking relationships among the parts. To a certain extent, the number of notes the rhythms have in common is less important than how the patterns emerge.

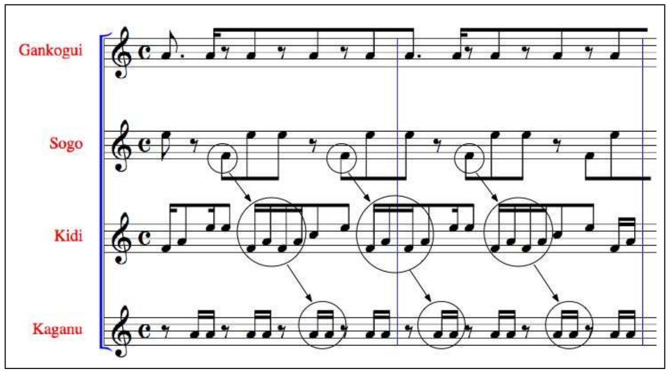

Figure 3. Interlocking rhythms in the *gahu* music.

Gamelan music is perhaps the best known musical tradition using interlocking rhythms. In Balinese classical music a gong-chime, or a set of tuned gongs called *réong*, is found in the *gamelan gong* [13]. Separate rhythmic patterns that interlock in a continuous figuration are performed on a pair of *réong* in contrast to the rhythm played by the soloist. Figure 4 shows the *réong* interplay.

Figure 4. Interlocking rhythms in Balinese gamelan.

The above musical examples inspire some further definitions of interlocking rhythms, assuming the same timespan. *Complementary interlocking rhythms* are a set of rhythms that have no common onset and exactly one onset on any pulse of the timespan. The *cáscara* rhythm, played with ghost notes and complementary canons, are an example of a complementary interlocking rhythm. *Disjoint interlocking rhythms* are interlocking rhythms that have no onset at a common position. Examples are the overture *Romeo and Juliet* and the *réong* parts from Balinese music. In order to define interlocking rhythms with common notes, like those present in *gahu* music, we must retain some constraints of rhythms being disjoint; otherwise every pair of rhythms would be interlocked. A family of rhythms is said to be *non-disjoint interlocking* if every rhythm in the family has no more than half of its onsets in common with any other rhythm in the same family. Note that if this definition is strictly applied, the three rhythms of *gahu* are not interlocking because only the note onsets are considered, while the salience produced by the texture is ignored. However, in a real performance of *gahu* the entire pattern would clearly emerge.

1.0.0.2. Euclidean Rhythms. The second family of rhythms we study here is the family of Euclidean rhythms, first introduced by Toussaint [18]. Roughly speaking, Euclidean rhythms are rhythms where the onsets are distributed among the pulses *as evenly as possible*. This idea of onsets spread out evenly among the pulses was first introduced by Clough and Myerson [8], and later expanded by Clough and Douthett [7]. Demaine et al. [9] have conducted a thorough study of Euclidean rhythms, their mathematical properties, and their connections with other fields, music in particular. They also describe three algorithms that generate Euclidean rhythms for given number of onsets and pulses, and show connections with Euclid's algorithm for finding the greatest common divisor of two integers.

In this paper we examine three operations on rhythms and study the conditions under which these operations, when applied to a Euclidean rhythm, produce other (one or more) Euclidean rhythms. The operations we study are: (1) complementation, (2) alternation, an operation that takes certain subsets of onsets from a rhythm to produce a new one, and (3) decomposition. We finally relate these operations to the definitions of interlocking rhythms.

2. Notation and Basic Definitions

A rhythm has many representations. In this paper we will use three main representations of rhythms. The first representation is the *box-like notation* [16], where onsets are represented with the symbol '×' and rests with the symbol '·'. This representation is equivalent to a binary string, where the 1-bits are the onsets of the rhythm and the 0-bits are the rests. The second is the *subset* notation, where the onsets are written as a subset of the n pulses numbered from 0 to $n-1$. The third representation used is the *clockwise distance sequence*, where the clockwise distance between a pair of consecutive onsets around the circular lattice is represented by an integer; these integers together sum to the total number of pulses. As an example, consider the Cuban clave *son* rhythm. Its three representations are [× · · × · · × · · · × · × · · ·] in box-like notation, $\{0, 3, 6, 10, 12\}$ in subset notation, and $(3, 3, 4, 2, 4)$ in clockwise distance sequence notation.

Let $E(k, n)$ denote the Euclidean rhythm with k onsets and n pulses generated by Bjorklund's algorithm [3] (to be described later), and let $E_{CD}(k, n)$ denote the Euclidean rhythm with k onsets and n pulses generated by the Clough-Douthett algorithm [9]. The onsets of $E_{CD}(k,n)$ are given by the sequence:

$$E_{CD}(k,n) = \left\{ \left\lfloor \frac{in}{k} \right\rfloor : i = 0, 1, \ldots, k-1 \right\}. \quad (1)$$

Note that this definition of E_{CD} is a more restricted form of the original algorithm given by Clough and Douthett; however, this restricted version is sufficient for our purposes in this paper.

3. Operations on Euclidean Rhythms

A musical piece may contain different rhythms and melodies throughout its progression. What musicians often like to do however is to stay within a theme during this progression. For example, a jazz soloist must respect the style and feeling of the piece, and thus play an improvised variation based on the foundation of the main theme. A way of realizing such an improvisation is by taking the base rhythm and transforming it to another through one or more *operations*. An operation transforms one musical rhythm to another based on a set of rules. It is generally desirable that this transformation preserves some properties of the original rhythm. Operations are thus important for music composition and especially for improvisation. They are also important for music analysis, for example, to understand rhythmic transformations. Therefore, studying operations is useful both for theoretical analysis as well as for providing formal rules for improvisation techniques. Here we study some properties of Euclidean rhythms under various operations, namely: complementation, alternation, and decomposition.

3.1. Complement of Euclidean Rhythms

The study of the complementary sets of sets of intervals in the context of pitch (scales and chords) has received a lot of attention in music theory [14]. The complement of rhythm, on the other hand, has scarcely been explored. Consider the Cuban *cinquillo* rhythm given by [× · ×× · ×× ·]. Its complementary rhythm is [· × · · × · · ×], which is a rotation of the famous Cuban *tresillo* rhythm given by [× · · × · · × ·].

Complementary sets have many applications, such as the composition of rhythmic complementary canons [19]. Clough and Douthett [7] show that the complement of a maximally even rhythm is maximally even. This theorem was later proved independently by other authors. Here we present a proof of the complementation theorem that is different from that of Clough and Douthet.

Theorem 3.1 [5, 7, 11] *The complement of a Euclidean rhythm is Euclidean up to a rotation.*

Proof: Harris and Reingold [11] showed that Euclid's algorithm for computing the greatest common divisor of two integers generates digital straight lines described by the Bresenham algorithm [4]. Euclid's algorithm also generates Euclidean rhythms [9]; thus, a sequence of 0-1 bits of a Euclidean rhythm corresponds to the sequence of 0-1 bits of a digital straight line (modulo rotation). Without loss of generality, assume that a 0-bit of a digital straight line corresponds to a vertical segment and a 1-bit corresponds to a horizontal segment.

Let R be a Euclidean rhythm corresponding to the digital line L defined by the equation $y = ax$ (we assume the line passes through the center of our coordinate system). If we rotate L by 90°, then the equation of the rotated line L' becomes $x = ay$. To draw the digital line L', we can merely interchange the x and y axis and plot line $L : y = ax$. This means that L and L' are the same digital line and hence are both described by Euclid's algorithm. However, when we rotate L, the 0-bits of R become 1-bits and the 1-bits become 0-bits. Hence we obtain the complement of R. Since both R and its complement correspond to the "same" digital line drawn by the same sequence of vertical and horizontal segments, they are both Euclidean. ■

COROLLARY 3.2 *The complement of a maximally even rhythm is maximally even.*

3.2. Alternation of Euclidean Rhythms

Starting from the j-th onset, the *alternation* operation keeps every c-th onset of a rhythm and transforms the remaining onsets into rests. More formally, let $\{u_0, u_1, \ldots, u_{k-1}\}$ be the sorted sequence of the positions of the k onsets of a rhythm R. The *j-alternation of order c* of R, denoted by $A_{j,c}(R)$, is the rhythm whose pulses are all rests except the onsets $u_j, u_{j+c}, u_{j+2c}, \ldots, u_{j+vc}$, for $0 \leq j < c \leq k-1$ and $v = \left\lfloor \frac{k-1-j}{c} \right\rfloor$.

In general alternations of a rhythm may not be rotation invariant. The alternation $A_{j,c}$ of a rhythm R and a rotation of R might not produce the same rhythm (or any of its rotations). To see this, consider the alternation $A_{0,2}$ of rhythms $E(7, 17)$ and $E_{CD}(7, 17)$, which are rotations of each other.

$$E(7,17) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot]$$

$$A_{0,2}(E(7,17)) = [\times \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \times \cdot]$$

$$E_{CD}(7,17) = [\times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot]$$

$$A_{0,2}(E_{CD}(7,17)) = [\times \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \times \cdot \cdot]$$

Clearly, the two alternations are not rotations of each other.

Euclidean rhythms fulfill very precise constraints on the durations of their onsets. By performing an alternation those durations are changed. In general, an arbitrary alternation might destroy the property of a rhythm being Euclidean. In this section we will determine under what conditions the alternation of a Euclidean rhythm is still Euclidean.

Note that when c does not divide k, we can obtain two alternations having different numbers of onsets by varying the value of j. Set $r = k \bmod c$; there are r alternations with $\lceil k/c \rceil$ onsets, which are $A_{0,c} \ldots, A_{r-1,c}$, and $c - r$ alternations with $\lfloor k/c \rfloor$ onsets, the remaining $A_{r,c} \ldots, A_{c-1,c}$. Indeed, we can write:

$$1 + \left\lfloor \frac{k-1-j}{c} \right\rfloor = 1 + \left\lfloor \frac{k-1-j+r-r}{c} \right\rfloor = 1 + \left\lfloor \frac{k-r}{c} + \frac{r-1-j}{c} \right\rfloor$$
$$= 1 + \left\lfloor \frac{k}{c} \right\rfloor + \left\lfloor \frac{r-1-j}{c} \right\rfloor.$$

If $j \leq r-1$, then $\left\lfloor \frac{r-1-j}{c} \right\rfloor$ is 0, and thus $1 + \left\lfloor \frac{k-1-j}{c} \right\rfloor = \left\lceil \frac{k}{c} \right\rceil$. When $r - 1 < j \leq c-1$, the quotient $\frac{r-1-j}{c}$ is negative, and $\left\lfloor \frac{r-1-j}{c} \right\rfloor = -1$, which implies that $1 + \left\lfloor \frac{k-1-j}{c} \right\rfloor = \left\lfloor \frac{k}{c} \right\rfloor$.

Consider the rhythm $E(8, 17)$ and its 3-alternations:

$$E(8,17) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot]$$
$$A_{0,3}(E(8,17)) = [\times \cdot \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot]$$
$$A_{1,3}(E(8,17)) = [\cdot \cdot \times \cdot \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \cdot \times \cdot \cdot]$$
$$A_{2,3}(E(8,17)) = [\cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \cdot \cdot]$$

Alternations $A_{0,3}(E(8, 17))$ and $A_{1,3}(E(8, 17))$ have $\lceil 8/3 \rceil = 3$ onsets, whereas alternation $A_{2,3}(E(8, 17))$ has $\lfloor 8/3 \rfloor = 2$ onsets. Here $r = 2 = 8 \bmod 3$, and therefore there are 2 alternations with 3 onsets and one alternation with 2 onsets. Note that $A_{2,3}(E(8, 17))$ is not Euclidean.

In the following, we make use of another characterization of Euclidean rhythms, that relates to the clockwise distances between onsets. Consider a rhythm R with k onsets and n pulses, represented as a set of points on a circular lattice of size n. If r_i and r_j are two onsets of R, then the *clockwise distance* of the ordered pair r_i, r_j is the length of the clockwise arc from r_i to r_j. Demaine et al. [9] (and previously Clough and Douthett [7]) showed the following:

LEMMA 3.3 **[9]** *A rhythm is Euclidean if and only if the clockwise distance of the ordered pair of onsets* $(r_i, r_{i+\ell})$ *is equal to either* $\left\lfloor \frac{\ell n}{k} \right\rfloor$ *or* $\left\lceil \frac{\ell n}{k} \right\rceil$ *for all* $\ell = 1, 2, \ldots, k$ *and* $i = 0, 1, \ldots, k-1$.

In particular, when $\ell = 1$ this characterization states that the clockwise distances between consecutive onsets take values of either $\left\lfloor \frac{n}{k} \right\rfloor$ or $\left\lceil \frac{n}{k} \right\rceil$. We can now show the following.

THEOREM 3.4 *The* j*-alternation of order* c *of any rotation of* $E(ck, n)$ *is Euclidean for* $c > 1$.

Proof: Let $E^*(ck, n)$ be a rotation of $E(ck, n)$. Denote by $r_i^{'}$ the onsets of $E^*(ck, n)$. Since $E^*(ck, n)$ is a rotation of $E(ck, n)$, there exist two integers $j \geq 0$ and m such that the formula $m + \left\lfloor \frac{n(i+j)}{ck} \right\rfloor$ generates the onsets of $E^*(ck, n)$. In addition, denote by r_i the onsets of $E(ck, n)$. Finally, let us compute the clockwise distances for the alternation.

Let ℓ be a positive integer.

$$r'_{i+\ell} - r'_i = m + \left\lfloor \frac{nj}{ck} + \frac{nc(i+\ell)}{ck} \right\rfloor - \left(m + \left\lfloor \frac{nj}{ck} + \frac{nci}{ck} \right\rfloor \right)$$
$$= r_{j+c(i+\ell)} - r_{j+ci} \in \left\{ \left\lfloor \frac{nc\ell}{ck} \right\rfloor, \left\lceil \frac{nc\ell}{ck} \right\rceil \right\} = \left\{ \left\lfloor \frac{n\ell}{k} \right\rfloor, \left\lceil \frac{n\ell}{k} \right\rceil \right\}$$

By Lemma 3.3, the last equality implies that the alternations of $E^*(ck, n)$ are rotations of $E(k, n)$. ■

LEMMA 3.5 *All the alternations of order 2 of $E_{CD}(k, n)$ are Euclidean if and only if k is even.*

Proof: First suppose k is even, and consider the two alternations $A_{0,2}(E_{CD}(k, n))$ and $A_{1,2}(E_{CD}(k, n))$. By Theorem 3.4, both of these alternations are Euclidean.

Now assume, for the sake of contradiction, that k is odd. By Lemma 3.3 the clockwise distance sequence of $E(k, n)$ is formed by only two distances, say a and b, where $b = a+1$ and $a \geq 1$. Moreover, the consecutive clockwise sequence of a Euclidean rhythm cannot have distances that differ by more than 1. The proof is divided into several cases:

(1) Rhythm $E(k, n)$ has the form $(\underbrace{b, b, \ldots, b}_{k-1}, c)$, where c is either a or b. When we keep the odd onsets, we obtain $(\underbrace{2b, 2b, \ldots, 2b}_{(k-1)/2}, c)$, which contains distances that differ by more than 1 (there is a distance equal to $2b - c > 1$). Thus, this alternation cannot be a Euclidean rhythm.

(2) Rhythm $E(k, n)$ has the form $(b, a, a, \ldots, a)$ (starts with b followed by a's). When we keep the odd onsets, we obtain $(\underbrace{b + a, 2a, \ldots, 2a}_{(k-1)/2}, a)$, which is not Euclidean (there is a distance equal to $a + b - a > 1$).

(3) Rhythm $E(k, n)$ has the form $(b, \ldots, a, \ldots, b)$ (starts and ends with b, having at least one a in between). If we take the odd onsets, the last distance wraps around and results in a distance of $2b + a$ or $3b$ in the alternation. On the other hand, the distance of a in $E(k, n)$ is transformed into a distance of either $2a$ or $a + b$ in the alternation. Hence, there is a pair of distances in the alternation that differs by more than 1 and, hence, the alternation cannot be Euclidean.

(4) Rhythm $E(k, n)$ is of the form $(\underbrace{a, a, \ldots, a}_{k-1}, b\}$. If we take the even onsets, we obtain a rotation of rhythm $(\underbrace{2a, \ldots, 2a}_{(k-1)/2}, 2a+b)$, which is not Euclidean (there is a distance equal to $2a + b - 2a > 1$). ■

Note that the proof of Theorem 3.5 is an argument on the clockwise distance sequence of $E_{CD}(k, n)$ and it can be applied to $E(k, n)$ or any of its rotations in a straightforward manner. Consequently, we state the following corollary.

COROLLARY 3.6 *All the alternations of order* 2 *of any rotation of* $E(k, n)$ *are Euclidean if and only if* k *is even.*

THEOREM 3.7 *Let* k *be such that* $1 \leq k < n$. *All the alternations of order* c *of* $E_{CD}(k, n)$ *are Euclidean if and only if* c *divides* k.

Proof: When c divides k, the fact that the alternation $A_{j,c}(E_{CD}(k, n))$ is Euclidean for all $j < c$ follows directly from Theorem 3.4.

Assume now for the sake of contradiction that c does not divide k and $A_{j,c}(k, n)$ is Euclidean for all $j \leq c$. Let $r = k \bmod c$; since c does not divide k, then $r \neq 0$. In this case there are r alternations having $\lceil k/c \rceil$ onsets and $c - r$ alternations having $\lfloor k/c \rfloor$ onsets. Consider two of these alternations, $A_{0,c}$ and $A_{r,c}$. Since $A_{0,c}$ is Euclidean by hypothesis, by Lemma 3.3 its consecutive clockwise distances can only belong to the set $\{\lfloor \frac{cn}{k} \rfloor, \lceil \frac{cn}{k} \rceil, \lfloor \frac{rn}{k} \rfloor$ or $\lceil \frac{rn}{k} \rceil\}$. This is because $A_{0,c}$ is formed by changing $c - 1$ onsets into rests from each consecutive block of c onsets, followed by a final block where only $r - 1$ onsets are changed into rests. The blocks of c onsets generate distances $\{\lfloor \frac{cn}{k} \rfloor, \lceil \frac{cn}{k} \rceil\}$, while the block of r onsets generates distances $\{\lfloor \frac{rn}{k} \rfloor, \lceil \frac{rn}{k} \rceil\}$. Consecutive clockwise distances of a Euclidean rhythm can only take on two values (Lemma 3.3), and therefore at least one of the following equalities must hold:

$$\left\lfloor \frac{cn}{k} \right\rfloor = \left\lfloor \frac{rn}{k} \right\rfloor, \left\lfloor \frac{cn}{k} \right\rfloor = \left\lceil \frac{rn}{k} \right\rceil, \left\lceil \frac{cn}{k} \right\rceil = \left\lfloor \frac{rn}{k} \right\rfloor, \text{or} \left\lceil \frac{cn}{k} \right\rceil = \left\lceil \frac{rn}{k} \right\rceil.$$

Since $r < c$, the first three equalities $\lfloor \frac{cn}{k} \rfloor = \lfloor \frac{rn}{k} \rfloor$, $\lceil \frac{cn}{k} \rceil = \lceil \frac{rn}{k} \rceil$ and $\lceil \frac{cn}{k} \rceil = \lfloor \frac{rn}{k} \rfloor$ lead to contradictions. Consider the equality $\lfloor \frac{cn}{k} \rfloor = \lceil \frac{rn}{k} \rceil$:

$$\left\lfloor \frac{cn}{k} \right\rfloor = \left\lceil \frac{rn}{k} \right\rceil \Rightarrow \left\lfloor (c - r + r)\frac{n}{k} \right\rfloor = \left\lfloor \frac{rn}{k} \right\rfloor + 1 \Rightarrow \left\lfloor (c - r)\frac{n}{k} + r\frac{n}{k} \right\rfloor = \left\lfloor \frac{rn}{k} \right\rfloor + 1$$
$$\Rightarrow \left\lfloor (c - r)\frac{n}{k} \right\rfloor + \left\lfloor \frac{rn}{k} \right\rfloor + \alpha = \left\lfloor \frac{rn}{k} \right\rfloor + 1 \Rightarrow \left\lfloor (c - r)\frac{n}{k} \right\rfloor + \alpha = 1,$$

where $\alpha = \lfloor \frac{(c-r) \bmod k + r \bmod k}{k} \rfloor$. If $\alpha = 1$, we obtain $\left\lfloor \frac{(c-r)n}{k} \right\rfloor = 0$, which lead to a contradiction. So, suppose $\alpha = 0$. Then, $\lfloor (c - r)\frac{n}{k} \rfloor = 1 \Longrightarrow 1 \leq (c - r)\frac{n}{k} < 2$. For this inequality to be true, $n < 2k$ and $r = c - 1$ must hold. This is the only case that, for the moment, does not cause a contradiction.

When considering $A_{r,c}$, the possible clockwise distances are $\{\lfloor \frac{cn}{k} \rfloor, \lceil \frac{cn}{k} \rceil, \lfloor \frac{(c+r)n}{k} \rfloor, \lceil \frac{(c+r)n}{k} \rceil\}$. The distances $\{\lfloor \frac{cn}{k} \rfloor, \lceil \frac{cn}{k} \rceil\}$ come from blocks of c onsets of $A_{r,c}$. The last block of $A_{r,c}$ wraps around and produces two blocks: one with c onsets located at the end of the rhythm, and another with r onsets located at the beginning. The distances for these blocks are $\{\lfloor \frac{(c+r)n}{k} \rfloor, \lceil \frac{(c+r)n}{k} \rceil\}$. Again, at least one of the following equalities must hold:

$$\left\lfloor \frac{cn}{k} \right\rfloor = \left\lfloor \frac{(c+r)n}{k} \right\rfloor, \left\lfloor \frac{cn}{k} \right\rfloor = \left\lceil \frac{(c+r)n}{k} \right\rceil, \left\lceil \frac{cn}{k} \right\rceil = \left\lceil \frac{(c+r)n}{k} \right\rceil, \text{or} \left\lceil \frac{cn}{k} \right\rceil = \left\lfloor \frac{(c+r)n}{k} \right\rfloor.$$

Since $c < c + r$, equalities $\lfloor \frac{cn}{k} \rfloor = \left\lfloor \frac{(c+r)n}{k} \right\rfloor$, $\lceil \frac{cn}{k} \rceil = \left\lceil \frac{(c+r)n}{k} \right\rceil$ and $\lfloor \frac{cn}{k} \rfloor = \left\lceil \frac{(c+r)n}{k} \right\rceil$ leads to a contradiction. Finally, consider equality $\left\lceil \frac{cn}{k} \right\rceil = \left\lfloor \frac{(c+r)n}{k} \right\rfloor$.

$$\left\lceil \frac{cn}{k} \right\rceil = \left\lfloor \frac{(c+r)n}{k} \right\rfloor = \left\lfloor c\frac{n}{k} + r\frac{n}{k} \right\rfloor \Rightarrow \left\lfloor \frac{cn}{k} \right\rfloor + 1 = \left\lfloor c\frac{n}{k} \right\rfloor + \left\lfloor r\frac{n}{k} \right\rfloor + \alpha \Rightarrow \left\lfloor r\frac{n}{k} \right\rfloor + \alpha = 1,$$

where $\alpha = \lfloor \frac{cn \bmod k + cr \bmod k}{k} \rfloor$, and hence α can only take the values 1 or 0. If $\alpha = 1$, then $\lfloor r\frac{n}{k} \rfloor = 0$, which gives a contradiction. Suppose $\alpha = 0$. In this case, $\lfloor r\frac{n}{k} \rfloor = 1$. This equality is true when $r = 1$ and $n < 2k$. Equality $r = 1$ contradicts equality $r = c - 1$ derived in the previous case, except for the value of $c = 2$. However, in Lemma 3.5 it was proved that this case cannot occur either. This concludes the proof. ■

Theorem 3.8 *All the alternations of order c of any rotation of $E(k,n)$ are Euclidean if and only if c divides k.*

Proof: Let $E^*(k,n)$ be a rotation of $E(k,n)$. There exist two integers $j_0 \geq 0$ and m such that the formula $m + \left\lfloor \frac{n(j_0+i)}{k} \right\rfloor$, $i = 0, \ldots, k-1$, outputs the onsets of $E^*(k,n)$. Furthermore, the onsets of the j-alternation of order c of $E^*(k,n)$ are given by the formula $m + \left\lfloor \frac{n(j_0+j+ic)}{k} \right\rfloor$ $i = 0, \ldots, k-1$. Denote by $r_i^{'}$ the onsets of $A_{j,c}(E^*(k,n))$ and by r_i those of $E(k,n)$. First, we will compute the distance between two consecutive onsets in $A_{0,c}(E^*(k,n))$ for any $i = 0, \ldots, \lfloor \frac{k}{c} \rfloor - 1$.

$$\begin{aligned} r_{i+1}^{'} - r_i^{'} &= m + \left\lfloor \frac{n(j_0 + (i+1)c)}{k} \right\rfloor - \left(m + \left\lfloor \frac{n(j_0 + ic)}{k} \right\rfloor \right) \\ &= r_{j_0+ci+c} - r_{j_0+ci} \in \left\{ \left\lfloor \frac{nc}{k} \right\rfloor, \left\lceil \frac{nc}{k} \right\rceil \right\} \end{aligned}$$

For the case $i = \lfloor \frac{k}{c} \rfloor$, i.e. for the distance between the last and the first onsets, we have:

$$\begin{aligned} r_{i+1}^{'} - r_i^{'} &= m + \left\lfloor \frac{n(j_0 + k)}{k} \right\rfloor - \left(m + \left\lfloor \frac{n(j_0 + \lfloor \frac{k}{c} \rfloor c)}{k} \right\rfloor \right) \\ &= m + \left\lfloor \frac{n(j_0 + k}{k} \right\rfloor - \left(m + \left\lfloor \frac{n(j_0 + k - r)}{k} \right\rfloor \right) \\ &= r_{j_0+k} - r_{j_0+k-r} \in \left\{ \left\lfloor \frac{nr}{k} \right\rfloor, \left\lceil \frac{nr}{k} \right\rceil \right\} \end{aligned}$$

We now look at the alternation $A_{r,c}(E^*(k,n))$. We keep denoting by $r_i^{'}$ the onsets of the alternation. For $i = 0, \ldots, \lfloor \frac{k}{c} \rfloor - 1$, the consecutive distances are:

$$\begin{aligned} r_{i+1}^{'} - r_i^{'} &= m + \left\lfloor \frac{n(j_0 + r + (i+1)c)}{k} \right\rfloor - \left(m + \left\lfloor \frac{n(j_0 + r + ic)}{k} \right\rfloor \right) \\ &= r_{j_0+r+ci+c} - r_{j_0+r+ci} \in \left\{ \left\lfloor \frac{nc}{k} \right\rfloor, \left\lceil \frac{nc}{k} \right\rceil \right\} \end{aligned}$$

For $i = \lfloor \frac{k}{c} \rfloor - 1$, we have:

$$r'_{i+1} - r'_i = m + \left\lfloor \frac{n(j_0 + r + k)}{k} \right\rfloor - \left(m + \left\lfloor \frac{n(j_0 + r + (\lfloor \frac{k}{c} \rfloor - 1)c)}{k} \right\rfloor \right)$$
$$= \left\lfloor \frac{n(j_0 + r + k)}{k} \right\rfloor - \left\lfloor \frac{n(j_0 + r + k - r - c)}{k} \right\rfloor$$
$$= r_{j_0+r+k} - r_{j_0+k-c} \in \left\{ \left\lfloor \frac{n(c+r)}{k} \right\rfloor, \left\lceil \frac{n(c+r)}{k} \right\rceil \right\}$$

For both alternations $A_{0,c}(E^*(k,n))$ and $A_{r,c}(E^*(k,n))$ we obtained the same set of consecutive distances: $\{\lfloor \frac{cn}{k} \rfloor, \lceil \frac{cn}{k} \rceil, \lfloor \frac{rn}{k} \rfloor, \lceil \frac{rn}{k} \rceil\}$ for $A_{0,c}(E^*(k,n))$ and $\{\lfloor \frac{cn}{k} \rfloor, \lceil \frac{cn}{k} \rceil, \lfloor \frac{(c+r)n}{k} \rfloor, \lceil \frac{(c+r)n}{k} \rceil\}$ for $A_{r,c}(E^*(k,n))$. The argument used in the proof of Theorem 3.7 repeats from this point on as the set of distances for the alternations are all the same. Thus, we finish the proof here. ■

3.3. Decomposition of Euclidean Rhythms

Let R_1 and R_2 be two rhythms with the same number of pulses represented as binary strings. The *union* of R_1 and R_2 is the rhythm R obtained by performing the logical *OR* operation between the ith bit of R_1 and the ith bit of R_2 for all $i = 0, 1, \ldots, n-1$. For example, the union of the two rhythms [× · · · ×× · × · · · ×] and [× · × · × · · × · × · ·] is the rhythm [× · × · ×× · × · × · ×]. We say that R can be *decomposed* into the two rhythms R_1 and R_2. A union of two rhythms R_1 and R_2 is *disjoint* if the bits in position i in R_1 and R_2 are not both 1 for any $0 \le i \le n-1$. We can think of the alternation operation as a form of rhythm decomposition. The following result follows from Theorem 3.4 from the previous section:

THEOREM 3.9 *Rhythm $E(k,n)$ can be decomposed into the disjoint union of d copies of $E(\frac{k}{d}, n)$ up to a rotation, where d is a divisor of k.*

The next result concerns the decomposition of a rhythm in terms of its complement.

THEOREM 3.10 *A Euclidean rhythm $E(k,n)$ with $\lfloor n/2 \rfloor < k \le n$ can be decomposed into the union of $\left\lfloor \frac{n}{n-k} \right\rfloor$ copies of the rotations of the rhythm $E(n-k,n)$. Such a union is disjoint if and only if $n-k$ divides n.*

Proof: Consider the complement $E^*(n-k,n)$ of the rhythm $E(k,n)$. We want to find a set of rhythms whose union has a 1-bit at every 0-bit position of $E^*(n-k,n)$. Let $E^*_i(n-k,n)$ denote the rhythm $E^*(n-k,n)$ rotated i steps in the clockwise direction, and consider the union R of the rhythms $E^*_1(n-k,n)$, $E^*_2(n-k,n)$, …, $E^*_x(n-k,n)$ where $x = \left\lfloor \frac{n}{n-k} \right\rfloor - 1$. Rhythm R has (possibly) fewer onsets than k, since $(n-k)\left(\left\lfloor \frac{n}{n-k} \right\rfloor - 1\right) = n - r - (n-k) = k - r \le k$, where $r = n \bmod (n-k)$. The number of consecutive onsets between pairs of 0-bits is exactly $\left\lfloor \frac{n}{n-k} \right\rfloor - 1$. By Lemma 3.3, the distance between two consecutive onsets in $E(n-k,n)$ is either $\left\lfloor \frac{n}{n-k} \right\rfloor$ or $\left\lceil \frac{n}{n-k} \right\rceil$. Hence, R has every 1-bit coinciding with a 0-bit position in $E^*(n-k,n)$. If $n-k$ divides n, then the distance between any two consecutive onsets in $E^*(n-k,n)$ is exactly $\frac{n}{n-k}$, and R has exactly k onsets; it follows that $R = E(k,n)$ and is the union of $\left\lfloor \frac{n}{n-k} \right\rfloor$ rhythms. If $n-k$ does not divide n, then the distance between any two consecutive onsets in $E^*(n-k,n)$ is either $\left\lfloor \frac{n}{n-k} \right\rfloor$ or $\left\lceil \frac{n}{n-k} \right\rceil$, and R has fewer than k onsets (and hence is not equal to any rotation of $E(k,n)$). The onsets of $E(k,n)$ that are missing in R are those between two consecutive onsets of $E^*(n-k,n)$ that are at a distance $\left\lceil \frac{n}{n-k} \right\rceil$. Let $E_{-1}(n-k,n)$ be the rotation of rhythm $E^*(n-k,n)$ by one step in the counterclockwise direction. The union of R and $E_{-1}(n-k,n)$ will add the missing onsets, and the rhythm resulting from this union will be the rhythm $E(k,n)$. Note that $R \cup E_{-1}(n-k,n)$ is not necessarily disjoint, and the total number of rhythms in this union is equal to $\left\lfloor \frac{n}{n-k} \right\rfloor - 1 + 1 = \left\lfloor \frac{n}{n-k} \right\rfloor$. ■

Theorem 3.11 *A Euclidean rhythm $E(k,n)$ with $\lfloor n/2 \rfloor < k < n$ is the union of rotations of two disjoint Euclidean rhythms $E(n-k,n)$ and $E(2k-n,n)$.*

Proof: Consider the rhythm $E(2(n-k),n)$. Applying Theorem 3.7 to $E(2(n-k),n)$, we can conclude that $E(2(n-k),n)$ is the disjoint union of rotations of $A_{0,2}(n-k,n)$ and $A_{1,2}(n-k,n)$. Moreover, by Theorem 3.4, $E(2(n-k),n)$ is the union of two copies of rotations of $E(n-k,n)$. On the other hand, $E(2(n-k),n)$ is the complement of $E(2k-n,n)$ up to a rotation, which implies that the two rhythms are disjoint. Therefore, by taking the complement of one of the alternations we obtain a decomposition of $E(k,n)$ into the disjoint union of two rotations of $E(n-k,n)$ and $E(2k-n,n)$. ■

4. Musical Connections

In this section we interpret the above mathematical results in musical terms. As mentioned at the outset, we are interested in the study of interlocking and Euclidean rhythms; in particular we will combine both ideas and study interlocking rhythms under the constraint that the resulting rhythm is Euclidean.

Let us start out by considering complementary interlocking rhythms formed by Euclidean rhythms. We already know that the complement of a Euclidean rhythm is Euclidean. Therefore, it is always possible to build pairs of complementary interlocking Euclidean rhythms. Next, as its most straightforward generalization, we may ask when a Euclidean rhythm can generate a tiling canon. According to Hall and Klingsberg [10], a tiling canon is a canon of periodic rhythms that has exactly one note onset (in some voice) per pulse. If $E(k,n)$ is a tiling canon, k necessarily divides n, and $c = \frac{n}{k}$ is the number of voices (or the number of times $E(k,n)$ is played). By applying our results on alternations (Theorems 3.4 and 3.9) to rhythm $E(n,n)$, we conclude that alternations $A_{1,c}(k,n), \ldots, A_{c,c}(k,n)$ are all clockwise rotations of $E(k,n)$ and that they tile a timespan of n pulses. It is worth noting that the resulting tiling canon is rather uninteresting musically. Because k divides n, rhythm $E(k,n)$ consists solely of an onset played regularly every $\frac{n}{k}$ pulses. The tiling canon is then composed of consecutive clockwise rotations of $E(k,n)$. For instance, consider a timespan of 12 pulses and a tiling canon of 4 onsets. The number of voices is $3 = 12/4$ and the tiling canon is:

$$A_{0,3} = [\times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot]$$
$$A_{1,3} = [\cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot]$$
$$A_{2,3} = [\cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times].$$

When the number of voices c does not divide n, tiling canons cannot be built. Let us generalize the concept of tiling canons so that this case can be subsumed. We define a *tiling quasi-canon* as a set of c periodic rhythms that tiles a timespan of n pulses and whose number of onsets for each pair of rhythms differs at most by one. Note that this definition includes that of usual tiling canons. Euclidean rhythms, even when c does not divide n, admit tiling quasi-canons. Indeed, the equality $n = c\lfloor n/c \rfloor + r$, where $r = n \bmod c$, can be rearranged as $n = (c-r)\lfloor n/c \rfloor + r\lceil n/c \rceil$. This equality implies that a timespan of n pulses can be tiled with $c-r$ rotations of $E(\lfloor n/c \rfloor, n)$ and r rotations of $E(\lceil n/c \rceil, n)$. It is enough to apply Theorem 3.4 to rhythm $E((c-r)\lfloor n/c \rfloor, n)$ and its complement $E(r\lceil n/c \rceil, n)$. For example, consider the problem of finding a tiling quasi-canon with 4 voices on a timespan of 19 pulses. Given that $r = 19 \bmod 4 = 3$, the tiling quasi-canon is formed by $c - r = 1$ rhythm of $\lfloor 19/4 \rfloor = 4$ onsets and $r = 3$ rhythms of $\lceil 19/4 \rceil = 5$ onsets, as shown below.

$$E(4,19) = [\times \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot]$$
$$E(5,19) = [\cdot \times \cdot \cdot \times \cdot \cdot \cdot \times \cdot \cdot \cdot \times \cdot \cdot \cdot \times \cdot \cdot]$$
$$E(5,19) = [\cdot \cdot \times \cdot \cdot \cdot \times \cdot \cdot \times \cdot \cdot \cdot \times \cdot \cdot \cdot \times \cdot]$$
$$E(5,19) = [\cdot \cdot \cdot \times \cdot \cdot \cdot \times \cdot \cdot \cdot \times \cdot \cdot \times \cdot \cdot \cdot \times]$$

Unlike the case of tiling canons, rotations of the tiling quasi-canon are not consecutive.

If we relax the complementarity constraint, we may consider disjoint interlocking rhythms. When $k < \lfloor n/2 \rfloor$ the decomposition of $E(k,n)$ is of no interest. Sometimes such a decomposition does not exist or is trivial (it is the union of rotations of $E(1,n)$). For example, $E(4,11) = [\times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot]$ cannot be decomposed into Euclidean rhythms with a smaller number of onsets, except for the trivial union of four copies of $E(1,11)$. Moreover, the interesting case is the decomposition of a dense rhythm in terms of sparser rhythms. Theorem 3.10 shows that, when $k > \lfloor n/2 \rfloor$, $E(k,n)$ can be decomposed into the union of two disjoint interlocking rhythms, namely, rotations of $E(n-k,n)$ and $E(2k-n,n)$. Below is an example of such a decomposition for rhythm $E(9,11)$.

$$E(9,11) = [\times \cdot \times \times \times \times \cdot \times \times \times \times]$$
$$E(7,11) = [\times \cdot \times \times \cdot \times \cdot \times \times \cdot \times]$$
$$E(2,11) = [\cdot \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \times \cdot]$$
$$\overline{E(9,11)} = [\cdot \times \cdot \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot]$$

Interestingly enough, rhythm $E(9,11)$ can also be decomposed into the union of rotations of rhythms $E(3,11)$ because 3 divides 9.

Note that this result also provides a new way for finding complementary interlocking rhythms. Since the complement of a Euclidean rhythm is Euclidean, for $k > \lfloor n/2 \rfloor$, there exists a decomposition of the timespan into the union of $E(2k-n, n)$ and two copies of $E(n-k, n)$, up to a rotation.

Finally, we analyze the union of Euclidean rhythms when this union is not required to be disjoint. Recall from the introduction that non-disjoint interlocking rhythms are defined as having no more than half of their onsets in common. Theorem 3.10 ensures that, when $k < \lfloor n/2 \rfloor$, there is a decomposition of $E(k, n)$ into not necessarily disjoint rhythms. If $n-k$ divides n, such a decomposition is disjoint; otherwise, the number of onsets in common is $n \bmod (n-k)$, a number that is less than $n-k < n/2$. For example, consider $E(9, 11)$.

$$E(9,11) = [\times \cdot \times\times\times\times \cdot \times\times\times\times]$$
$$E(2,11) = [\cdot \cdot \times \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot]$$
$$E(2,11) = [\cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \times \cdot \cdot]$$
$$E(2,11) = [\cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \times \cdot]$$
$$E(2,11) = [\cdot \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \times]$$
$$E(2,11) = [\times \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \cdot]$$

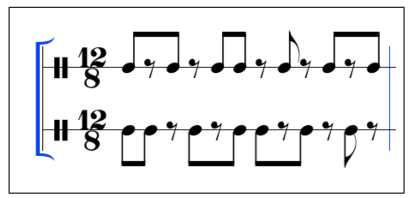

Figure 5. The bembé timeline and one of its rotations.

As a tangible musical example, let us consider the bembé, a style from the Afro-Cuban musical tradition. Its instrumentation includes at least one bell that plays a timeline also called bembé $[\times \cdot \times \cdot \times\times \cdot \times \cdot \times \cdot \times]$, which is a rotation of the Euclidean rhythm $E(7, 12)$. Sometimes in the bembé ensemble there are two bell players, one playing the bembé itself, or also known as the standard pattern, and the other playing a rotation of it [6, 17], as illustrated in Figure 5. These two rhythms are interlocking Euclidean rhythms given that they have only two onsets in common, those at positions 1 and 6.

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