Orderly Broadcasting in a 2D Torus

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Abstract

In this paper, we describe an ordering of the vertices of a 2-dimensional torus and study the upper bound on the orderly broadcast time. Along with messy broadcasting, orderly broadcasting is another model where the nodes of the network have limited knowledge about their local neighborhood. However, while messy broadcasting explores the worst-case performance of broadcast schemes, orderly broadcasting, like the classical broadcast model, is concerned with finding a fixed ordering of the vertices of a graph that will minimize the overall broadcast time.

1. Introduction

The term broadcasting refers to the process of information dissemination in a communication network whereby a message, originating from one of the nodes, is transmitted to all the other nodes in the network. A communication network can be modeled as an undirected graph. Thus, during broadcasting a given vertex needs to disseminate its message to all the remaining vertices of the graph.

Substantial effort has been made recently to make this dissemination efficient and fast. Various broadcasting models have been introduced [2, 5, 11, 10, 7], which deal with the issue of finding a scheme whereby this dissemination process takes the least amount of time. In these broadcast models, known as *classical broadcasting*, it was assumed that each vertex has the knowledge of the graph topology, the originator of the message and the time at which it was sent. Based on this information, and in order to minimize the broadcast time, each vertex of the graph transmits the message in the most *clever* way.

In 1994, Ahlswede *et al* [1] introduced a model of broadcasting called *messy broadcasting*. Unlike the classical model, messy broadcasting dealt with analyzing the worst-case performance of broadcast schemes. But the major difference between messy broadcasting and the classical broadcast model is that in the messy broadcast scheme, the vertices know nothing about the topology, and at each broadcast round transmit the message to a randomly selected neighbor.

In practice however, it is not realistic to require for each node of a network to know the network topology or to make decisions based on a set of stored protocols. In many cases, the nodes of the network have primitive structures with small memories that cannot store such information or make intelligent decisions. On the other hand, building networks in which the nodes have no decision-making responsibility is much simpler and more robust. These were the reasons that made the study of messy broadcasting interesting.

In messy broadcasting, when a vertex u receives a message, it randomly chooses a neighbor at each round and sends the message to that chosen neighbor. Ultimately, the neighbors of vertex u will be informed in some order. This means that if we randomly number the neighbors of each vertex $u \in V$ and let u send the message first to the neighbor numbered 1, then 2, ... etc. then we would have simulated an instance of messy broadcasting. The overall broadcast time of the graph in this case will depend on the way in which the neighbors of each vertex are numbered. A different ordering of the numbers might yield a different broadcast time; and the question here is : what is the ordering that will give the minimum possible broadcast time for a given graph?

The above question has lead to the idea of *orderly broadcasting* [4, 9] , which deals with the problem of finding an ordering for each vertex u of a given graph G that will minimize the overall broadcast time of G.

In what follows, we will state a more formal definition of the orderly broadcast model and give known bounds on some studied graphs.

2. Definitions and Notations

We will model a communication network as a connected graph $G = (V, E)$ consisting of a set $V = \{v_0, v_1, \dots v_n\}$ of vertices (nodes) and a set of edges E (communication lines) connecting these vertices. Two vertices $u \in V$ and $v \in V$ are *adjacent* or are *neighbors* when they are connected by an edge $e \in E$ such that $e = (u, v)$. We define the degree of a vertex v, $\delta(v)$, to be the number of neighbors of vertex v. A *path* P of a graph G is a sequence of vertices of the form $P = (v_1, v_2, \dots v_k)$ $(k \geq 2)$, such that every $(v_i, v_{i+1}) \in E$, $1 \leq i \leq k$. A graph $G = (V, E)$ is said to be *connected* if for every pair of vertices $u, v \in V$ there exists a path that connects u to v. The *distance*, $d(u, v)$, between two vertices $u \in V$ and $v \in V$ is the number of edges of the shortest path between u and v. The *diameter* $D(G)$ of a graph G is the maximum distance between all pairs of vertices of G.

The *originator* is a vertex $v \in V$ which initiates broadcasting by transmitting its message to the remaining vertices through the edges. Broadcasting is complete when all the vertices of the graph are informed. It is required that broadcasting is completed as fast as possible and subject to the following constraints:

i) each transmission requires one time unit.

ii) a vertex can transmit only to an adjacent vertex.

iii) a vertex can transmit the message to one vertex in one time unit.

Given an originator $u \in V$, we define the broadcast time of vertex u, $b(u)$ in the classical model to be the minimum number of time units required to complete broadcasting from vertex u [2, 5]. The classical broadcast time of a graph $G = (V, E)$ is:

$$b(G) = max\{b(u) | u \in V\}$$

Since each vertex can inform one of its adjacent vertices in one round, then at each time unit the number of informed vertices can at most be doubled. Thus, after k time units, the number of informed vertices is bounded by 2^k. A trivial lower bound for broadcasting a message in any graph will be $\lceil log_2(|V|) \rceil$.

The messy broadcast time of a vertex $u \in V$, $b^m(u)$ is the maximum number of time units required to complete broadcasting from vertex u [1, 8, 3]. The messy broadcast time of a graph $G = (V, E)$ is:

$$b^m(G) = max\{b^m(u) | u \in V\}$$

In this paper, $b^m(G)$ refers to the messy broadcast time of G under the model M_3. In this model, every informed vertex knows to which neighbors it has sent the message and will transmit it to the neighbors to which it has not yet sent, if any, in each time unit.

In [8], Harutyunyan and Liestman obtain exact values for the messy broadcasting time of complete graphs, cycles, complete d-ary trees, and hypercubes. Later, Comellas *et al* [3] continue the study of messy broadcasting and obtain exact values and bounds on the messy broadcast times of multidimensional directed tori.

A *symmetric digraph* is a graph $G = (V, E)$ where every edge $(u, v) \in E$ in the undirected graph becomes two edges (u, v), (v, u) in the digraph. We define the *outedge* of a vertex u to be the edge $(u, v) \in E$. Broadcasting in a symmetric digraph is identical to broadcasting in the associated undirected graph. Hence, in all further discussion we will model our communication network as a symmetric digraph.

An ordering Π_u of a vertex $u \in V$ of degree d is the assignment of distinct time units $1, 2, \dots d$, called *labels*, to the outedges of u. We use $\Pi(u, v)$ to denote the label assigned to the edge from u to v. When every vertex u of a graph G has an ordering Π_u, we say that G has ordering Π.

Suppose $G = (V, E)$ is a graph with ordering Π and let $u \in V$ such that $v_1, v_2, \dots, v_k \in V$ are the neighbors of u. Orderly broadcasting from originator u proceeds as follows: At time 0, the originator u learns the message. Vertex v_1 learns the message at time $\Pi(u, v_1)$; v_2 learns the message at time $\Pi(u, v_2)$; v_k learns the message at time $\Pi(u, v_k)$. In general, for any vertex $u \in V$, if u learns the message at time t, it sends the message along its outedge ordered i at time $t + i$. Orderly broadcasting is complete when every vertex has received the message. The time at which this happens is denoted by $b^\Pi(u)$. The broadcast time of graph $G = (V, E)$ with ordering Π is :

$$b^\Pi(G) = max\{b^\Pi(u) | u \in V\}$$

The orderly broadcast time of graph G is the minimum broadcast time over all possible orderings Π of the vertices of G. That is,

$$b^o(G) = min_\Pi\{b^\Pi(G)\}$$

Define $b^\Pi(P_i, v)$ to be the time at which vertex $v \in V$ receives the message through path P_i under ordering Π. If $u \in V$ and $v \in V$ are adjacent such that $k = \Pi(u, v)$, then $u \xrightarrow{k} v$ implies that if u receives the message at time t, then it will send it to v at time $t + k$.

The orderly broadcast time is known only for a limited number of graphs. Below we present some of the results described in [4, 9] :

1. For any tree T, $b^o(T) \leq b(T) + \left\lceil \frac{D(T)}{2} \right\rceil$

2. For a path P_n on n vertices, $b^o(P_n) = \left\lceil \frac{3n - 4}{2} \right\rceil$.

3. For a cycle C_n on n vertices, $b^o(C_n) = \left\lceil \frac{2n}{3} \right\rceil$.

4. For a complete graph K_n on n vertices, $b^o(K_n) \leq \lceil logn \rceil + 2 \lceil \sqrt{logn} \rceil$

5. For a grid $G_{m \times n}$ on mn vertices, $b^o(G_{m \times n}) = m + n - 1$

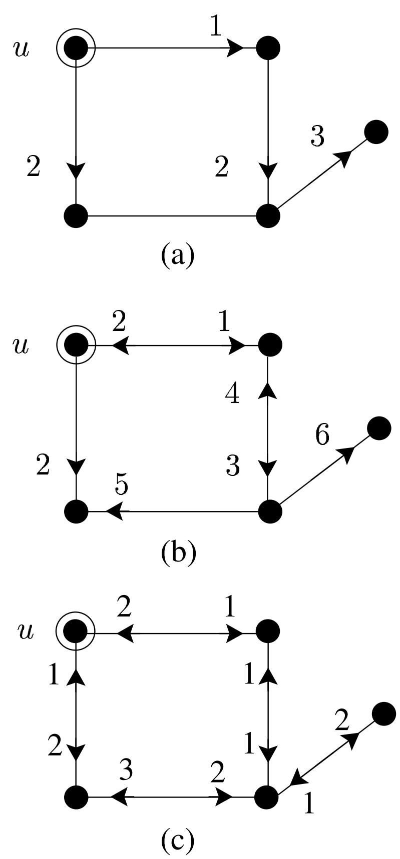

Figure 1. The three broadcast models: (a) Classical broadcasting with $b(u) = 3$ (b) Messy broadcasting with $b^m(u) = 6$ and (c) Orderly broadcasting with $b^o(u) = 4$. Note that in (c) the numbers represent the ordering of the edges and not the time at which the vertex receives the message.

In this paper, we describe an ordering Π for 2-dimensional tori and study the orderly broadcast time obtained for this ordering.

A 2-dimensional torus $T_{m \times n} = (V, E)$ with m rows and n columns ($m \leq n$) is a connected graph of mn vertices and $2mn$ edges, such that:

$$V(T_{m \times n}) = \{(i,j) | 0 \leq i \leq m-1 \text{ and } 0 \leq j \leq n-1\}$$
$$E(T_{m \times n}) = \{((u,v),(p,q)) | p = u \pm 1 \bmod m \text{ or } q = v \pm 1 \bmod n / (u,v) \in V, (p,q) \in V\}$$

The diameter of a 2D Torus is:

$$D(T) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$$

$$\text{Let } \ell = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor & \text{if } \left\lfloor \frac{n}{2} \right\rfloor \text{ is even} \\ \left\lfloor \frac{n}{2} \right\rfloor + 1 & \text{if } \left\lfloor \frac{n}{2} \right\rfloor \text{ is odd} \end{cases}$$

Here we make the following observation:

Observation 2.1

$b(G) \leq b^o(G) \leq b^m(G)$ *for any connected graph* G.

In the following sections we will first describe the ordering Π of a 2-dimensional torus $T_{m \times n}$ and discuss the upper bound for broadcasting under Π.

3. An Ordering for $T_{m \times n}$

In this section we define an ordering Π for $T_{m \times n}$. Observe that $T_{m \times n}$ is a 4-regular graph, hence the time units assigned to the edges will be chosen from the set of labels $\{1, 2, 3, 4\}$.

Let $u = (i,j) \in V, v \in V$ and $k \in Z^+$. Then,

1. $\Pi(u,v) = 1$ if:
   a) $v = (i, j+1 \bmod n)$ where $i = 2k$ and $j \notin \{0, \ell\}$
   b) $v = (i, j-1 \bmod n)$ where $i = 2k+1$ and $j \notin \{0, \ell\}$
   c) $v = (i+1 \bmod m, j)$ where $j = 0$
   d) $v = (i-1 \bmod m, j)$ where $j = \ell$

2. $\Pi(u,v) = 2$ if:
   a) $v = (i+1 \bmod m, j)$ where $j = 2k$ and $j \notin \{0, \ell\}$
   b) $v = (i-1 \bmod m, j)$ where $j = 2k+1$ and $j \notin \{0, \ell\}$
   c) $v = (i, j+1 \bmod n)$ where $i = 2k$ and $j \in \{0, \ell\}$
   d) $v = (i, j-1 \bmod n)$ where $i = 2k+1$ and $j \in \{0, \ell\}$

3. $\Pi(u,v) = 3$ if:
   a) $v = (i-1 \bmod m, j)$ where $j = 2k$ and $j \notin \{0, \ell\}$
   b) $v = (i+1 \bmod m, j)$ where $j = 2k+1$ and $j \notin \{0, \ell\}$
   c) $v = (i, j-1 \bmod n)$ where $i = 2k$ and $j \in \{0, \ell\}$
   d) $v = (i, j+1 \bmod n)$ where $i = 2k+1$ and $j \in \{0, \ell\}$

4. $\Pi(u,v) = 4$ if:
   a) $v = (i, j-1 \bmod n)$ where $i = 2k$ and $j \notin \{0, \ell\}$
   b) $v = (i, j+1 \bmod n)$ where $i = 2k+1$ and $j \notin \{0, \ell\}$
   c) $v = (i-1 \bmod m, j)$ where $j = 0$
   d) $v = (i+1 \bmod m, j)$ where $j = \ell$

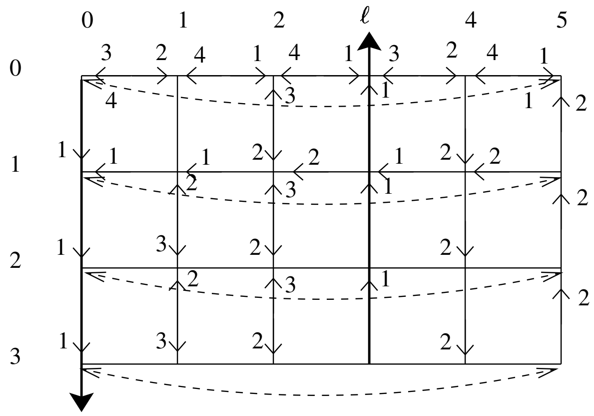

Figure 2. The ordering Pi for $T_{4\times 6}$. Note that some of the edges and labels are omitted to make the figure more readable.

In what follows we will assume that $T_{m\times n}$ has the ordering Π described above (Figure 2).

4. Upper Bound

Theorem 4.1 *A torus $T_{m\times n}$ with an ordering Π will have the following broadcast time :*

$$b^{\Pi}(T_{m\times n}) \le \begin{cases} D(T)+5 & \text{if } m \text{ and } n \text{ are odd} \\ D(T)+4 & \text{if } (m+n \text{ is odd}) \text{ or } \left(\dfrac{m}{2} \text{ and } \dfrac{n}{2} \text{ are both even}\right) \\ D(T)+3 & \text{otherwise} \end{cases}$$

The proof of the above theorem is omitted because of space restrictions. However, in the following we will show in detail how a message propagates through the vertices of a given row of $T_{m\times n}$.

Consider the torus $T_{8\times 13}$. The diameter $D(T_{8\times 13}) = 10$ and $\ell = 7$. Let the originator be vertex $(0,3)$, and suppose we want to inform all the vertices on row 5.

We describe four labeled paths :

- P_1 and P_2, that will inform every vertex $(5,j) \in V$ such that $0 < j \le 7$, and
- P_3 and P_4, that will inform every vertex $(5,j) \in V$ such that $7 < j \le 12$ and $j = 0$.

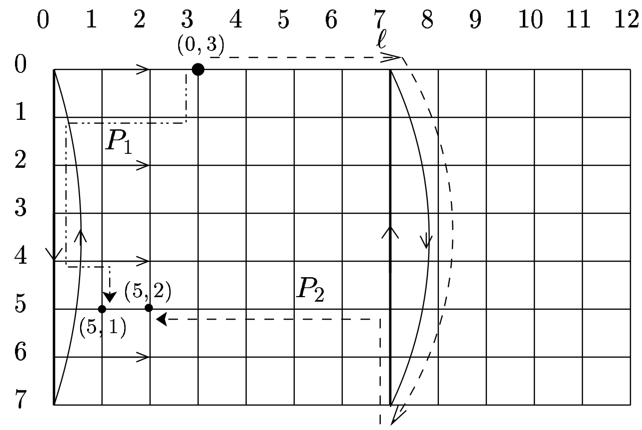

Figure 3. The two paths P_1 and P_2 originating from $(0,3)$ inform all the vertices $(5,j)$ such that $0 < j \le 7$.

First, we concentrate on informing all the vertices $(5,j)$ where $0 \le j \le 7$. Consider the paths P_1 and P_2 (Figure 3):

$$P_1 : (0,3) \xrightarrow{3} (1,3) \xrightarrow{1} (1,2) \xrightarrow{1} (1,1) \xrightarrow{1} (1,0) \xrightarrow{1} (2,0) \xrightarrow{1} (3,0) \xrightarrow{1} (4,0) \xrightarrow{2} (4,1) \xrightarrow{3} (5,1)$$

$$P_2 : (0,3) \xrightarrow{1} (0,4) \xrightarrow{1} (0,5) \xrightarrow{1} (0,6) \xrightarrow{1} (0,7) \xrightarrow{1} (7,7) \xrightarrow{1} (6,7) \xrightarrow{1} (5,7) \xrightarrow{2} (5,6) \xrightarrow{1} (5,5) \xrightarrow{1} (5,4) \xrightarrow{1} (5,3) \xrightarrow{1} (5,2)$$

The time that vertices $v_1 = (5,1)$ and $v_2 = (5,2)$ receive the message from each of the paths P_1 and P_2 are: $b^{\Pi}(P_1, v_1) = 14$ and $b^{\Pi}(P_2, v_2) = 13$

Now, we look at the vertices $(5,j)$ where $7 \le j \le 12$ Consider the two paths (Figure 4):

$$P_3 : (0,3) \xrightarrow{3} (1,3) \xrightarrow{1} (1,2) \xrightarrow{1} (1,1) \xrightarrow{1} (1,0) \xrightarrow{1} (2,0) \xrightarrow{1} (3,0) \xrightarrow{1} (4,0) \xrightarrow{1} (5,0) \xrightarrow{2} (5,12) \xrightarrow{1} (5,11)$$

$$P_4 : (0,3) \xrightarrow{1} (0,4) \xrightarrow{1} (0,5) \xrightarrow{1} (0,6) \xrightarrow{1} (0,7) \xrightarrow{1} (7,7) \xrightarrow{1} (6,7) \xrightarrow{2} (6,8) \xrightarrow{1} (6,9) \xrightarrow{1} (6,10) \xrightarrow{3} (5,10)$$

The time that vertices $v_3 = (5,11)$ and $v_4 = (5,10)$ receive the message from each of the paths P_1 and P_2 are: $b^{\Pi}(P_3, v_3) = 13$ and $b^{\Pi}(P_4, v_4) = 13$

Therefore, all the vertices of row 5, except vertices $(5,8)$ and $(5,9)$, will be informed through the above described paths P_1, P_2, P_3, P_4 in at most $D(T) + 4 = 14$ time units.

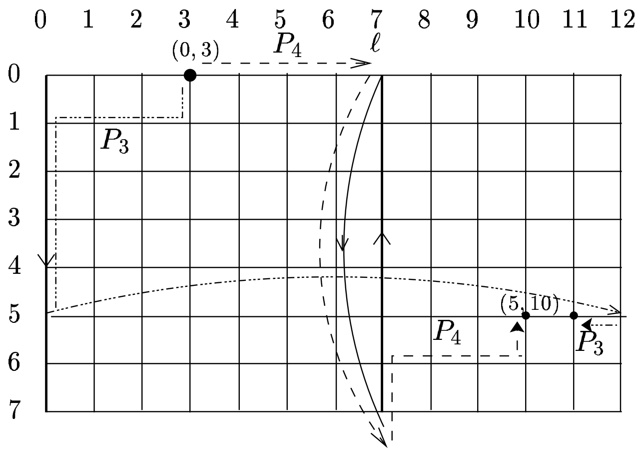

Figure 4. The two paths P_3 and P_4 originating from $(0, 3)$ inform all the vertices $(5, j)$ such that $7 < j \leq 12$ and $j = 0$.

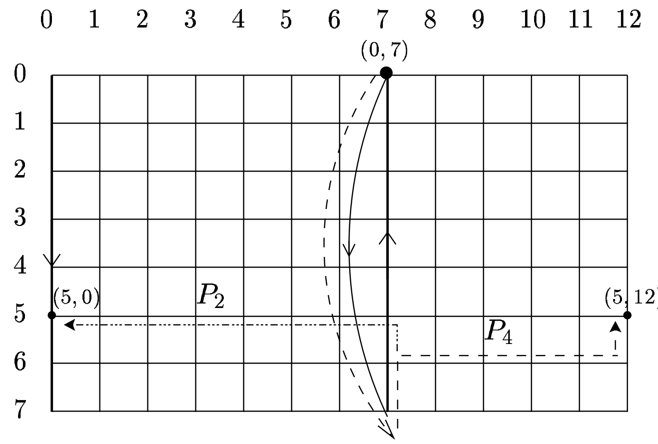

Figure 5. When the originator is $(0, 7)$, the paths P_2 and P_4 inform all the vertices of $T_{8\times 13}$ by time 11

Vertex $(5, 8)$ will be informed through the path:
$(0, 3) \xrightarrow{1} (0, 4) \xrightarrow{1} (0, 5) \xrightarrow{1} (0, 6) \xrightarrow{1} (0, 7) \xrightarrow{1} (7, 7)$
$\xrightarrow{1} (6, 7) \xrightarrow{2} (6, 8) \xrightarrow{3} (5, 8)$

Vertex $(5, 9)$ will be informed through the path:
$(0, 3) \xrightarrow{1} (0, 4) \xrightarrow{1} (0, 5) \xrightarrow{1} (0, 6) \xrightarrow{1} (0, 7) \xrightarrow{1} (7, 7)$
$\xrightarrow{1} (6, 7) \xrightarrow{2} (6, 8) \xrightarrow{1} (6, 9) \xrightarrow{2} (5, 9)$

From the above path descriptions it is easy to see that both vertices $(5, 8)$ and $(5, 9)$ will also receive the message in less than 14 time units.

Thus, we can conclude that in a $T_{8\times 13}$, for a given originator $(0, 3)$, all the vertices $v = (5, j) \in V$ on row 5 will be informed by time $b^{\Pi}(v) \leq D(T) + 4 = 14$
Similarly, we can show that the remaining rows of $T_{8\times 13}$ will be informed by time 14.

It can also be shown that for any other originator $u \in V$, all the vertices of $T_{8\times 13}$ will be informed in at most 14 time units. Figure 5 shows the paths used to disseminate the message in $T_{8\times 13}$ when the originator is $(0, 7)$. In fact, it can be proved that the worst-case for the originator (i, j) (when at least one of m or n is even) will be when j is odd - which is the case we described for originator $(0, 3)$.

5. Conclusion

In this paper we described an ordering of a 2-dimensional torus that will give an orderly broadcast time of at most $D(T) + 4$.

Recall that the classical broadcast time of a 2-dimensional torus is (see [6]) :

$$b(T_{m\times n}) \leq \begin{cases} D(T) & \text{if } m \text{ and } n \text{ are even} \\ D(T) + 1 & \text{otherwise} \end{cases}$$

On the other hand, the messy broadcast time (under the model M_3) for directed tori is:

$$b^m(T_{m\times n}) \leq 2(n-1) + (m-1)$$

Note that the above result is for special kinds of directed tori which are different from the one considered in this paper.

Compared to the results mentioned above, our result on the orderly broadcast time of a 2D torus is close to the broadcast time of a 2D torus in the classical model. An obvious lower bound on $b^o(T_{m\times n})$ is $D(T)$. Thus, our upper bound on $b^o(T)$ is quite tight.

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