Convergence of the shadow sequence of inscribed polygons

Francisco Gomez-Martin* Perouz Taslakian† Godfried T. Toussaint†

1 Introduction

Let P be a polygon inscribed in a circle. The *shadow* of P is a polygon P' whose vertices are at the midpoints of the arcs of consecutive points of P. The *shadow sequence* $P^0, P^1, P^2, \ldots$ is a sequence of inscribed polygons such that each P^t is the shadow of P^{t-1} for all $t \geq 1$. We show in this abstract that the shadow sequence converges to the regular polygon, and in such way that the variance of the edge lengths decreases by at least one half at every step. Our proof extends to the more general case where instead of placing the vertices of the shadow at the ratio of $1/2$ of every arc we place them at an arbitrary fixed ratio α ($0 < \alpha < 1$) in the clockwise or counterclockwise direction.

Sequences of polygons generated by performing iterative processes on an initial polygon have been studied extensively in geometry. One of the most studied sequences is the one sometimes referred to as Kasner polygons. Given a polygon P^0, the Kasner descendant P^1 of P^0 is obtained by placing the vertices of P^1 at the midpoints of the edges of P^0. Fejes Tóth [6] was interested in the more general problem of sequences of Kasner polygons where each polygon P^t in the sequence is obtained by dividing every edge of P^{t-1} with a ratio $\alpha : (1-\alpha)$ in the clockwise (or counterclockwise) direction and making the division points the vertices of P^t for $t = 1, 2, \ldots$. He proved that if $\alpha = 1/2$ (Kasner polygon), then the sequence converges to a regular polygon when P^0 is a convex pentagon or a convex hexagon. He conjectured that for any α and any initial convex polygon, the sequence converges to a regular polygon. Reichardt [5] showed that if $\alpha = 1/2$, every convex polygon converges to the regular polygon. Later, Lükő [4] proved that for any $\alpha \in [0, 1]$ and for any convex polygon P^0, the sequence $P^0, P^1, P^2, \ldots$ converges to a regular polygon, thus settling the more general conjecture of Fejes Tóth. More on Kasner polygons can be found in [2, 1].

The shadow sequence we study in this abstract is similar to the Kasner sequence. It is a sequence of inscribed polygons, where vertices of P^t are at midpoints of the *arcs* between consecutive vertices of P^{t-1}, for all $t \geq 0$. Hitt and Zhang [3] show that the shadow sequence of any inscribed polygon converges to a regular polygon. From their proof, it follows that the area of each P^t is greater than or equal to the area of P^{t-1} for any $t > 0$, with equality only when P^t is regular. In their proof of the convergence of the shadow sequence, Hitt and Zhang make use of doubly stochastic matrices and Schur-convex functions. In the next section, we give a simpler proof of the convergence of the shadow sequence that uses a different approach and is more intuitive. This proof also gives a bound on the rate of convergence. We then show how our results extend to the general shadow sequence.

2 Convergence of the Shadow

Let $P^0, P^1, \ldots$ be a sequence of inscribed polygons where P^t is the shadow of P^{t-1} for all $t \geq 0$. Then,

Theorem 1 *The shadow sequence of an inscribed polygon converges to a regular polygon in such a way that the variance of the edge lengths decreases at each step by at least one half.*

Proof: Let P be a polygon inscribed in a unit circle, and let $\langle a_0, a_1, \ldots, a_{n-1} \rangle$ be the sequence of edge lengths of P, where by *length* we mean the geodesic distance along the circle between two consecutive vertices of P. Thus, $\sum_{i=0}^{n-1} a_i = 1$. Let P^t denote the polygon P after t shadow operations, and a_i^t denote its corresponding edge lengths,

*Departamento de Matemática Aplicada, Univ. Politécnica de Madrid, Madrid, Spain, fmartin@eui.upm.es

†School of Computer Science, CIRMMT, McGill University, {perouz,godfried}@cs.mcgill.ca

for $i = 0, 1, \ldots, n-1$. At any step t, we can write the edge lengths of P^{t+1} as the sequence: $\left\{ \frac{a_i^t + a_{i+1}^t}{2} : i = 0, 1, \ldots, n-1 \right\}$. Also, the average edge length of P^t is $1/n$ for any t. Since the edge lengths sum to 1 at any step, we can treat the sequence of edges as a random variable and compute its variance. We will show that the variance V^{t+1} of the sequence of edge lengths at time $t+1$ decreases by a constant fraction of the variance at time t. For simplicity, we will assume $a_i^t = a_i$; thus,

$$
\begin{aligned}
& V^{t+1} \\
= \;& \frac{1}{n} \sum_{i=0}^{n-1} \left(a_i^{t+1}\right)^2 - \frac{1}{n^2} \\
= \;& \frac{1}{n} \sum_{i=0}^{n-1} \left(\frac{a_i + a_{i+1}}{2}\right)^2 - \frac{1}{n^2} \\
= \;& \frac{1}{n} \sum_{i=0}^{n-1} \frac{a_i^2 + a_{i+1}^2}{4} + \frac{1}{n} \sum_{i=0}^{n-1} \frac{a_i a_{i+1}}{2} - \frac{1}{n^2} \\
= \;& \frac{1}{2n} \sum_{i=0}^{n-1} a_i^2 - \frac{1}{2n^2} + \frac{1}{2n} \sum_{i=0}^{n-1} a_i a_{i+1} - \frac{1}{2n^2} \\
= \;& \frac{1}{2} \left(\frac{1}{n} \sum_{i=0}^{n-1} a_i^2 - \frac{1}{n^2} \right) + \frac{1}{2n} \sum_{i=0}^{n-1} a_i a_{i+1} - \frac{1}{2n^2} \\
= \;& \frac{1}{2} V^t + \frac{1}{2n} \sum_{i=0}^{n-1} a_i a_{i+1} - \frac{1}{2n^2}.
\end{aligned}
$$

To show that V^{t+1} is at most a fraction of V^t at any step t, we find the maximum value of $\sum_{i=0}^{n-1} a_i a_{i+1}$ subject to the constraint $\sum_{i=0}^{n-1} a_i = 1$. Using Lagrange multipliers it can easily be determined that the maximum value of the above sum is attained when all the a_i's have the same value, namely $1/n$. To find this maximal point, we solve:

$$
\frac{\partial}{\partial a_j} \left(\sum_{i=0}^{n-1} a_i a_{i+1} \right) + \lambda \left(\sum_{i=0}^{n-1} a_i - 1 \right) = 0
$$

Differentiating these n equations (for $j = 0, 1, \ldots, n-1$) we get $a_{i-1} + a_{i+1} + \lambda = 0$. Using the constraint $\sum_{i=0}^{n-1} a_i = 1$ we find that $a_i = \frac{1}{n}$. Thus, we have:

$$
\begin{aligned}
V^{t+1} &= \frac{1}{2} V^t + \frac{1}{2n} \sum_{i=0}^{n-1} a_i a_{i+1} - \frac{1}{2n^2} \\
&< \frac{1}{2} V^t + \frac{1}{2n} \sum_{i=0}^{n-1} \frac{1}{n} \cdot \frac{1}{n} - \frac{1}{2n^2} = \frac{1}{2} V^t
\end{aligned}
$$

Therefore, after every shadow step the variance is at least halved; and since the variance is always bounded below by zero, then it converges to zero as t goes to infinity. This in turn implies that every edge length converges to the mean value, which is $1/n$. Thus, the shadow sequence of any inscribed polygon converges to the regular polygon in such a way that the variance decreases by at least $1/2$ at every step.□

Let us now consider the general case, where each polygon P^{t+1} in the shadow sequence is obtained by dividing every arc of P^t with a ratio $\alpha : (1-\alpha)$ $(0 < \alpha < 1)$ in the clockwise (or counterclockwise) direction, and making the division points the vertices of P^{t+1} for all $t \geq 0$. In this case, at any step t we can write the edge lengths of P^{t+1} as the sequence: $\left\{ (1-\alpha) a_i^t + \alpha a_{i+1}^t : i = 0, 1, \ldots, n-1 \right\}$. We can extend the proof of Theorem 1 to show that the variance of the edges of the generalized shadow sequence decreases at every step by at least $2\alpha^2 - 2\alpha + 1$. By doing calculations similar to those made in the proof of Theorem 1, we can show that $V^{t+1} < (2\alpha^2 - 2\alpha + 1) V^t$. Note that $0 < 2\alpha^2 - 2\alpha + 1 < 1$ for any $0 < \alpha < 1$. Therefore, after every shadow step the variance decreases by a constant fraction that depends on α. Thus, again the variance converges to zero as t goes to infinity, and every edge length converges to $1/n$.

References

[1] F. Bachmann and E. Schmidt. *N-gons*. Univ. of Toronto Press, 1975. Trans. by C.W.L. Garner.

[2] P. J. Davis. *Circulant Matrices*. Chelsea Pub Co, 1994. 2nd edition.

[3] R. Hitt and X.-M. Zhang. Dynamic geometry of polygons. *Elemente der Math.*, 56:21–37, 2001.

[4] G. Lükõ. Certain sequences of inscribed polygons. *Periodica Mathematica Hungarica*, 3(3–4):255–260, 1973.

[5] H. Reichardt. Bestätigung einer vermutung von Fejes Tóth. *Revue Roumaine de Mathématiques Pures et Appliquées*, 15:1513–1518, 1970.

[6] L. F. Tóth. Iteration methods for convex polygons. *Mathematikai Lapok*, 20:15–23, 1969.